QUESTION 3 BINARY SYSTEM

Solution

a) The total angular momentum of the system is

$$L = I\omega = (M_1 r_1^2 + M_2 r_2^2)\omega \qquad \text{(0.5 points)}$$

We have also $M_1 r_1 = M_2 r_2$ and $D = r_1 + r_2$

which yield $$L = \frac{M_1 M_2}{M_1 + M_2} D^2 \omega \quad \ldots\ldots\ldots\ldots\ldots\ldots(1) \qquad \text{(0.5 points)}$$

The kinetic energy of the system is

$$K.E. = \frac{1}{2}M_1\left(r_1\omega\right)^2 + \frac{1}{2}M_2\left(r_2\omega\right)^2 = \frac{1}{2}\left(M_1 r_1^2 + M_2 r_2^2\right)\omega^2 \qquad \text{(0.5 points)}$$

$$= \frac{1}{2}\frac{M_1 M_2}{(M_1 + M_2)} D^2 \omega^2 \quad \ldots\ldots\ldots\ldots\ldots(2) \qquad \text{(0.5 points)}$$

b) From Newton's laws of motion we have

$$M_1 \omega^2 r_1 = M_2 \omega^2 r_2 = \frac{G M_1 M_2}{D^2} \qquad \text{(1 point)}$$

These equations together with those in a) yield

$$\omega^2 = \frac{G\left(M_1 + M_2\right)}{D^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3) \qquad \text{(1 point)}$$

c) In order to find the quantity $\Delta\omega$, we must also realize that

$$M_1 + M_2 = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots(4) \qquad \text{(0.5 points)}$$

And that, since there is no external torque acting on the system, the total angular momentum must be conserved.

$$L = \frac{M_1 M_2}{M_1 + M_2} D^2 \omega = \text{constant}$$

that is, $$M_1 M_2 D^2 \omega = \text{constant} \qquad \text{(0.5 points)}$$

Now, after the mass transfer,

$$\begin{aligned} \omega &\to \omega + \Delta\omega \\ M_1 &\to M_1 + \Delta M_1 \\ M_2 &\to M_2 - \Delta M_1 \\ D &\to D + \Delta D \end{aligned}$$

Hence, $M_1M_2D^2\omega = (M_1+\Delta M_1)(M_2-\Delta M_1)(D+\Delta D)^2(\omega+\Delta\omega)$

After using the approximation $(1+x)^n \sim 1+nx$ and rearranging, we get

$$\frac{\Delta\omega}{\omega}+2\frac{\Delta D}{D} = \left(\frac{M_1-M_2}{M_1M_2}\right)\Delta M_1 \quad \text{..........(5)} \qquad \text{(1 point)}$$

From equation (3), ω^2D^3 is also constant. That is,

$$\omega^2D^3 = (\omega+\Delta\omega)^2(D+\Delta D)^3$$

This gives,

$$\frac{\Delta D}{D} = -\frac{2}{3}\frac{\Delta\omega}{\omega} \quad \text{..................(6)} \qquad \text{(0.5 points)}$$

Hence $$\Delta\omega = -\frac{3(M_1-M_2)}{M_1M_2}\omega\,\Delta M_1 \quad \text{..........(7)} \qquad \text{(0.5 points)}$$

d) Given that

$M_1 = 2.9\ M_\odot$, $M_2 = 1.4\ M_\odot$

orbital period, $T = 2.49$ days

and that T has increased by 20 s in the past 100 years,

we have $\omega = \frac{2\pi}{T}$ and $\Delta\omega = -\frac{\omega}{T}\Delta T$ ....(8) (0.5 points)

$$\Delta M_1 = +\frac{1}{3}\left(\frac{M_1M_2}{M_1-M_2}\right)\frac{\Delta T}{T} \qquad \text{(0.5 points)}$$

$$\frac{\Delta M_1}{M_1\Delta t} = \frac{1}{3}\left(\frac{1.4}{(2.9-1.4)}\right)\left(\frac{20}{2.49\times 24\times 3600}\right)\left(\frac{1}{100}\right)$$

$$= 2.89\times 10^{-7} \text{ per year} \qquad \text{(0.5 points)}$$

e) Mass is flowing from M_2 to M_1. (0.5 points)

f) From equations (6) and (8):

$$\frac{\Delta D}{D\Delta t} = -\frac{2}{3}\frac{\Delta\omega}{\omega} = +\frac{2}{3}\frac{\Delta T}{T} = 6.20\times 10^{-7} \text{ per year} \quad \text{(1.0 points)}$$