QUESTION 4 GRAVITATIONAL LENSING

Solution

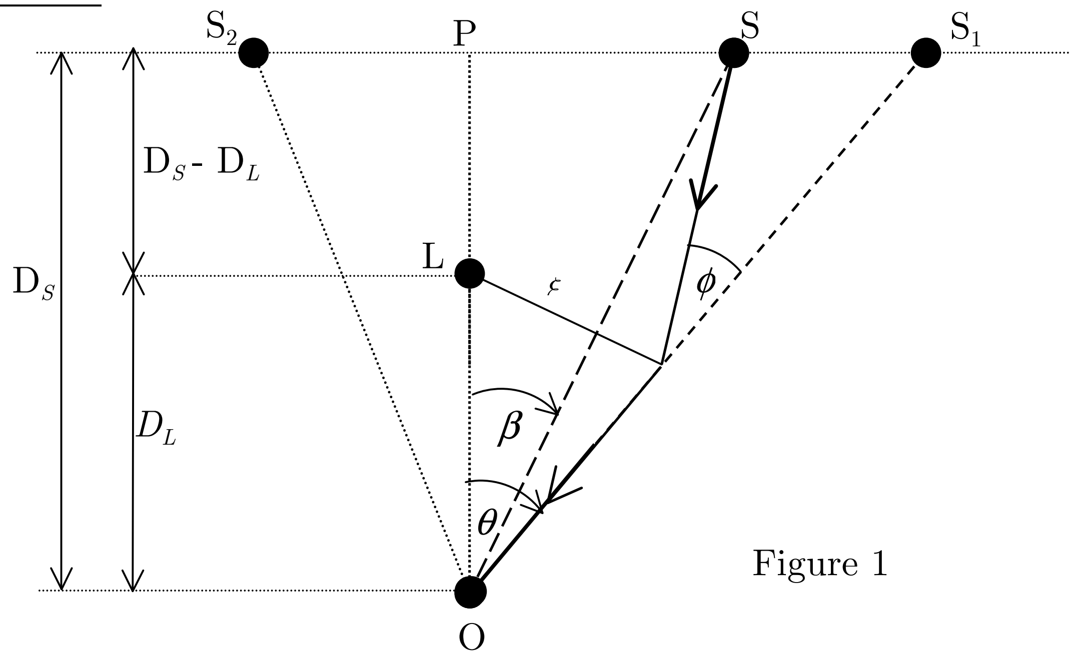

Figure 1

a). From figure 1, for small angles, $\tan\theta \approx \theta$, hence

$$\begin{aligned} PS_1 &= PS + SS_1 \\ \theta D_S &= \beta D_S + \left(D_S - D_L\right)\phi \\ \theta - \beta &= \frac{4GM}{\xi c^2}\frac{\left(D_S - D_L\right)}{D_S} \quad \ldots\ldots\ldots\ldots(1) \end{aligned}$$

(1 point)

Note that $\theta = \dfrac{\xi}{D_L}$ also, hence,

$$\theta^2 - \beta\theta = \left(\frac{4GM}{c^2}\right)\left(\frac{D_S - D_L}{D_L D_S}\right) \quad \ldots\ldots\ldots\ldots(2)$$

For a perfect alignment in which $\beta = 0$, we have $\theta = \pm\theta_E$, where

$$\theta_E = \sqrt{\left(\frac{4GM}{c^2}\right)\left(\frac{D_S - D_L}{D_L D_S}\right)} \quad \ldots\ldots\ldots\ldots(3)$$

(1 point)

b). From equation (3), for a solar-mass lens with $D_S = 50$ kpc,

$D_L = 50 - 10 = 40$ kpc

$$\theta_E = \sqrt{\left(\frac{4GM}{c^2}\right)\left(\frac{D_S - D_L}{D_L D_S}\right)} = 0.956 \times 10^{-9} \text{ radian}$$

$$= 1.97 \times 10^{-4} \text{ arc second}$$ (1 point)

c.) The resolution of the Hubble space telescope having diameter of 2.4 m is,

$$\theta_{Hubble} = 1.22\frac{\lambda}{D} = 1.22 \times \frac{5 \times 10^{-7}\text{ m}}{2.4\text{ m}} = 2.54 \times 10^{-7} \text{ radian}$$ for light of

wavelength 500 nm. (1 point)

Hence the Hubble telescope could not resolve this Einstein ring.

(1 point)

d). The quadratic equation (2) has two distinct roots, namely,

$$\theta_1 = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \theta_E^2}$$ (1 point)

$$\theta_2 = \frac{\beta}{2} - \sqrt{\left(\frac{\beta}{2}\right)^2 + \theta_E^2}$$ (1 point)

where $\theta_E = \sqrt{\left(\frac{4GM}{c^2}\right)\left(\frac{D_S - D_L}{D_L D_S}\right)}$

This implies that there are two images for a single isolated source.

e).

$$\frac{\theta_{1,2}}{\beta} = \frac{\frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 + \theta_E^2}}{\beta}$$

$$= \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\eta}\right)^2} = \frac{1}{2}\left[1 \pm \frac{\sqrt{\eta^2 + 4}}{\eta}\right]$$ (1 point)

f). From equation (2) $\theta^2 - \beta\theta - \theta_E^2 = 0$

$$(\theta + \Delta\theta)^2 - (\beta + \Delta\beta)(\theta + \Delta\theta) - \theta_E^2 = 0$$

$$\frac{\Delta\theta}{\Delta\beta} = \frac{\theta}{2\theta - \beta}$$ (1 point)

$$\left(\frac{\Delta\theta}{\Delta\beta}\right)_{\theta=\theta_{1,2}} = \frac{\theta_{1,2}}{2\theta_{1,2}-\beta} = \frac{\frac{1}{2}\eta \pm \sqrt{1+\frac{\eta^2}{4}}}{\pm 2\sqrt{1+\frac{\eta^2}{4}}} = \frac{1}{2}\left[1 \pm \frac{\eta}{\sqrt{\eta^2+4}}\right] \quad \text{(1 point)}$$