Theoretical Competition

Please read these instructions carefully:

1. Each student will receive problem sheets in English and/or in his/her native language.
2. The available time for answering theoretical problems is 5 hours. You will have 15 short problems (Theoretical Part 1, Problem 1 to 15), and 2 long problems (Theoretical Part 2, Problem 16 and 17).
3. Use only the pen that is provided on your desk.
4. Do **Not** use the back side of your writing sheets. Write only inside the boxed area.
5. Yellow scratch papers are not considered in marking.
6. Begin answering each problem in separate sheet.
7. Fill in the boxes at the top of each sheet of your paper with your "country name", your "student code", "problem number", and total number of pages which is used to answer to that problem.
8. **Write the final answer for each problem in the box, labeled "Answer Sheet".**
9. Starting and the end of the exam will be announced by ringing a bell.
10. The final answer in each question must be accompanied by units, which should be in SI or appropriate units as specified in the problem. 20% of the marks available for that part will be deducted for a correct answer without units.
11. The required numerical accuracy for the final answer depends on the number of significant figures given in the data values in the problem. 20% of the marks available for the final answer in each question part will be deducted for answers without required accuracy as given in the problem. Use the constant values exactly as given in the table of constants.
12. At the end of the exam put all papers, including scratch papers, inside the envelope and leave everything on your desk.

Table of Constants

(All constants are in SI)

Parameter	Symbol	Value
Gravitational constant	G	$6.67 \times 10^{-11} N\, m^2\, kg^{-2}$
Plank constant	h	6.63×10^{-34} J s
	$\hbar$	$1.05 \times 10^{-34}\ J\, s$
Speed of light	c	$3.00 \times 10^{8}\ m\, s^{-1}$
Solar Mass	$M_{\odot}$	$1.99 \times 10^{30}\ kg$
Solar radius	$R_{\odot}$	$6.96 \times 10^{8}\ m$
Solar luminosity	$L_{\odot}$	$3.83 \times 10^{26}\ W$
Apparent solar magnitude (V)	$m_{\odot}$	-26.8
Solar constant	$b_{\odot}$	$1.37 \times 10^{3}\ W\, m^{-2}$
Mass of the Earth	$M_{\oplus}$	$5.98 \times 10^{24}\ kg$
Radius of the Earth	$R_{\oplus}$	$6.38 \times 10^{6}\ m$
Mean density of the Earth	$\rho_{\oplus}$	$5 \times 10^{3}\ kg\, m^{-3}$
Gravitational acceleration at sea level	g	$9.81\ m\, s^{-2}$
Tropical year		365.24 days
Sidereal year		365.26 days
Sidereal day		86164 s
Inclination of the equator with respect to the ecliptic	ε	23.5°
Parsec	pc	$3.09 \times 10^{16}\ m$
Light year	ly	$9.46 \times 10^{15}\ m$
Astronomical Unit	AU	$1.50 \times 10^{11}\ m$
Solar distance from the center of the Galaxy	R	$8 \times 10^{3} pc$
Hubble constant	H	$75\ kms^{-1}\ M\, pc^{-1}$
Mass of electron	m_e	$9.11 \times 10^{-31}\ kg$
Mass of proton	m_p	$1.67 \times 10^{-27}\ kg$
Central wavelength of V-band	λ	$550\ n\, m$
Refraction of star light at horizon		$34'$
	π	3.1416

Useful mathematical formula:

$$\ln(1+x) \sim x \qquad for\ \ x \to 0$$

Short Problems: (10 points each)

Problem 1: Calculate the mean mass density for a super massive black hole with total mass of 1×10^8 $M_\odot$ inside the Schwarzschild radius.

Problem 2: Estimate the number of photons per second that arrive on our eye at $\lambda = 550\ nm$ (V-band) from a G2 main sequence star with apparent magnitude of $m = 6$ (the threshold of naked eye visibility). Assume the eye pupil diameter is $6\ mm$ and all the radition from this star is in $\lambda = 550\ nm$.

Problem 3: Estimate the radius of a planet that a man can escape its gravitation by jumping vertically. Assume density of the planet and the Earth are the same.

Problem 4: In a typical Persian architecture, on top of south side windows there is a structure called "Tabeshband" (shader), which controls sunlight in summer and winter. In summer when the Sun is high, Tabeshband prevents sunlight to enter rooms and keeps inside cooler. In the modern architecture it is verified that the Tabeshband saves about 20% of energy cost. Figure (1) shows a vertical section of this design at latitude of $36°.0$ N with window and Tabeshband.

Using the parameters given in the figure, calculate the maximum width of the Tabeshband, "x", and maximum height of the window , "h" in such a way that:

i) No direct sunlight can enter to the room in the summer solstice at noon.
ii) The direct sunlight reaches the end of the room (indicated by the point **A** in the figure) in the winter solstice at noon.

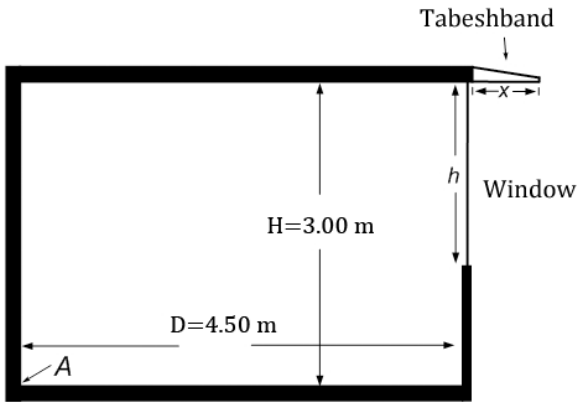


Figure (1)

Problem 5: The Damavand Mountain is located at the North part of Iran, in south coast of Caspian Sea. Consider an observer standing on the Damavand mountain top (latitude = $35°57'$N; longitude = $52°6'$E; altitude $5.6 \times 10^3\ m$ from the mean sea level) and looking at the sky over the Caspian Sea. What is the minimum declination for a star, to be seen marginally circumpolar for this observer. Geodetic radius of the Earth at this latitude is $6370.8\ km$. Surface level of the Caspian Sea is approximately equal to the mean sea level.

Problem 6: Derive a relation for the escape velocity of an object, launched from the center of a proto-star cloud. The cloud has uniform density with the mass of M and radius R. Ignore collisions between the particles of the cloud and the launched object. If the object were allowed to fall freely from the surface, it would reach the center with a velocity equal to $\sqrt{\frac{GM}{R}}$.

Problem 7: A student tries to measure field of view (FOV) of the eyepiece of his/her telescope, using rotation of the Earth. To do this job, the observer points the telescope towards Vega (alpha Lyr., RA: 18.5^h, Dec: $+39°$), turns off its "clock drive" and measures trace out time, t=5.3 minutes, that Vega crosses the full diameter of the FOV. What is the FOV of this telescope in arc-minutes?

Problem 8: Estimate the mass of a globular cluster with the radius of $r = 20\, pc$ and root mean square velocity of stars equal to $v_{rms} = 3\ kms^{-1}$.

Problem 9: The Galactic longitude of a star is $l = 15°$. Its radial velocity with respect to the Sun is $V_r = 100\ kms^{-1}$. Assume stars in the disk of the Galaxy are orbiting the center with a constant velocity of $V_0 = 250\ kms^{-1}$ in circular orbits in the same sense in the galactic plane. Calculate distance of the star from the center of the Galaxy.

Problem 10: A main sequence star with the radius and mass of $R = 4R_\odot$, $M = 6M_\odot$ has an average magnetic field of 1×10^{-4} T. Calculate the strength of the magnetic field of the star when it evolves to a neutron star with the radius of $20\ km$.

Problem 11: Assume the mass of neutrinos is $m_\nu = 10^{-5}m_e$. Calculate the number density of neutrinos (n_ν) needed to compensate the dark matter of the universe. Assume the universe is flat and 25 % of its mass is dark matter. Hint: Take the classical total energy equal to zero

Problem 12: Calculate how much the radius of the Earth's orbit increases as a result of the Sun losing mass due to the thermo-nuclear reactions in its center in 100 years. Assume the Earth's orbit remains circular during this period.

Problem 13: Assume that you are living in the time of Copernicus and do not know anything about Kepler's laws. You might calculate Mars-Sun distance in the same way as he did. After accepting the revolutionary belief that all the planets are orbiting around the Sun, not around the Earth, you measure that the orbital period of Mars is 687 days, then you observe that 106 days after opposition of Mars, the planet appears in quadrature. Calculate Mars-Sun distance in astronomical unit (AU).

Problem 14: A satellite is orbiting around the Earth in a circular orbit in the plane of the equator. An observer in Tehran at the latitude of $\varphi = 35.6^{\circ}$ N observes that the satellite has a zenith angle of $z = 46.0°$, when it transits the local meridian. Calculate the distance of the satellite from the center of the Earth (in the Earth radius unit).

Problem 15: An eclipsing close binary system consists of two giant stars with the same sizes. As a result of mutual gravitational force, stars are deformed from perfect sphere to the prolate spheroid with $a = 2b$, where a and b are semi-major and semi-minor axes (the major axes are always co-linear). The inclination of the orbital plane to the plane of sky is $90°$. Calculate the amplitude of light variation in magnitude (Δm) as a result of the orbital motion of two stars. Ignore temperature variation due to tidal deformation and limb darkening on the surface of the stars. Hint: A prolate spheroid is a geometrical shape made by rotating of an ellipse around its major axis, like rugby ball or melon.

Long Problems:

Problem 16: High Altitude Projectile (45 points)

A projectile which initially is put on the surface of the Earth at the sea level is launched with the initial speed of $v_{\circ} = \sqrt{(GM/R)}$ and with the projecting angle (with respect to the local horizon) of $\theta = \frac{\pi}{6}$. M and R are the mass and radius of the Earth respectively. Ignore the air resistance and rotation of the Earth.

a) Show that the orbit of the projectile is an ellipse with a semi-major axis of $a = R$.
b) Calculate the highest altitude of the projectile with respect to the Earth surface (in unit of Earth radius).
c) What is the range of the projectile (distance between launching point and falling point)?
d) What is eccentricity (e) of the ellipse?
e) Find the flying time for the projectile.

Problem 17: Apparent number density of stars in the Galaxy (45 points)

Let us model the number density of stars in the disk of Milky Way Galaxy with a simple exponential function of $n = n_0 \exp\left(-\frac{r-R_0}{R_d}\right)$, where r represents the distance from the center of the Galaxy, R_0 is the distance of the Sun from the center of the Galaxy, R_d is the typical size of disk and n_0 is the stellar density of disk at the position of the Sun. An astronomer observes the center of the Galaxy within a small field of view. We take a particular type of Red giant stars (red clump) as the standard candles for the observation with approximately constant absolute magnitude of $M = -0.2$,

a) Considering a limiting magnitude of $m = 18$ for a telescope, calculate the maximum distance that telescope can detect the red clump stars. For simplicity we ignore the presence of interstellar medium so there is no extinction.

b) Assume an extinction of $0.70\ mag/kpc$ for the interstellar medium. Repeat the calculation as done in the part (a) and obtain a rough number for the maximum distance these red giant stars can be observed.

c) Give an expression for the number of these red giant stars per magnitude within a solid angle of Ω that we can observe with apparent magnitude in the range of m and $m + \Delta m$, (i.e. $\frac{\Delta N}{\Delta m}$). Red giant stars contribute f of overall stars. In this part assume no extinction in the interstellar medium as part (a).

Hint : the Tylor expansion of $y = \log_{10} x$ is :

$$y = y_0 + \frac{1}{ln10}\frac{x - x_0}{x}$$

Solutions

Solution 1:

Schwarzschild radius of a black hole with mass M is

$$R = \frac{2GM}{c^2}$$ (4 points)

Then the mass density can be estimated as

$$\rho = \frac{M}{\frac{4}{3}\pi\frac{8G^3M^3}{c^6}} = \frac{3c^6}{32\pi}\frac{1}{G^3M^2}$$ (2 points)

$$= \frac{3\times(3.00\times10^8)^6}{32\times3.1416}\frac{1}{(6.67\times10^{-11})^3(10^8\times1.99\times10^{30})^2}$$

$$= 1.85\times10^3\ kg\ m^{-3}$$ (4 points)

Solution 2:

To calculate flux of a $m = 6$ star we use the Sun as standard candle

$$m_1 - m_2 = -2.5\log\frac{f_1}{f_2}$$ (1 point)

$$6 - (-26.8) = -2.5\log\frac{f_1}{1.37\times10^3}$$

$$f_1 = 1.04\times10^{-10}(w/m^2)$$ (3 points)

We need to know how much energy a visual photon has (at 550 nm)

$$E_p = h\nu = \frac{hc}{\lambda}$$

$$= \frac{6.63 \times 10^{-34} \times 3.00 \times 10^{8}}{550 \times 10^{-9}}$$

(3 points)

$$= 3.62 \times 10^{-19}\, J$$

Then number of photon which arrive to our eye per second is

(3 points)

$$N = \frac{f_1 \pi r_e^2}{E_p} = \frac{1.04 \times 10^{-10}}{3.62 \times 10^{-19}} \times 3.1416 \times 0.003^2 = 8 \times 10^3 \;\; s^{-1}$$

Solution 3:

We must compare the jumping speed of a normal human with escape velocity of the planet .
A normal human can jump up to 50 cm then his initial velocity is

$$v = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 0.5}$$
$$v = 3.13\, ms^{-1}$$

(3 points)

Comparing this velocity with escape velocity of the planet

$$v = \sqrt{\frac{2GM}{R}}$$

(3 points)

$$v^2 = \frac{2GM}{R} = \frac{2G}{R}\frac{4\pi}{3}\rho\, R^3$$

$$= \frac{8\pi G}{3}\rho\, R^2$$ (2 points)

$$R^2 = \frac{3v^2}{8\pi G\rho}$$

$$R = 2 \times 10^3\ m$$ (2 points)

Solution 4:

The zenith angle of the sun at summer solstice will be

$$z_s = \phi - 23.5 = 12.5$$ (1.5 points)

And in the winter solstice

$$z_w = \phi + 23.5 = 59.5$$ (1.5 points)

Figures shows that in summer solstice we have

$$\tan(z_s) = \frac{x}{h} = 0.22$$ (1.5 points)

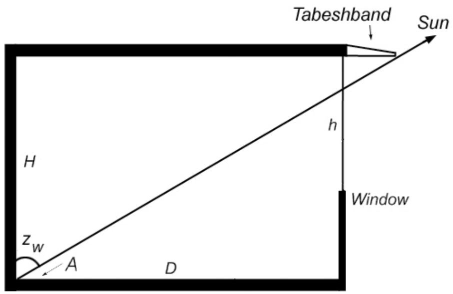
And in the winter solstice

$$\tan(z_w) = \frac{D + x}{H} = 1.70$$ (1.5 points)

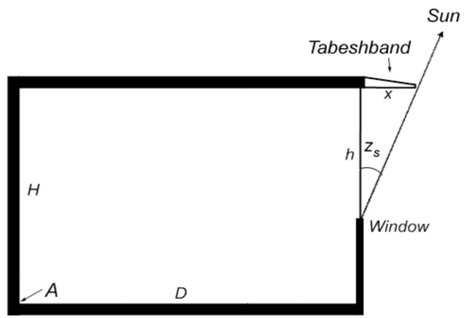
Then

$$x = 1.70H - D = 0.60\ m$$ (2 points)

$$h = 2.73\ m$$ (2 points)



Winter solstice



Summer solstice

Solution 5:

To calculate accurate value for minimum declination for circumpolar stars two major effect must be considered.

1. Refraction in earth atmosphere, which is 34′ at horizon. (3 points)
2. Horizon depression which is

$$\mathrm{Cos}\,\theta = \frac{R}{R+h} = \frac{6370.8}{6370.8+5.6} \Longrightarrow \theta = 2^\circ 24'$$ (3 points)

Then

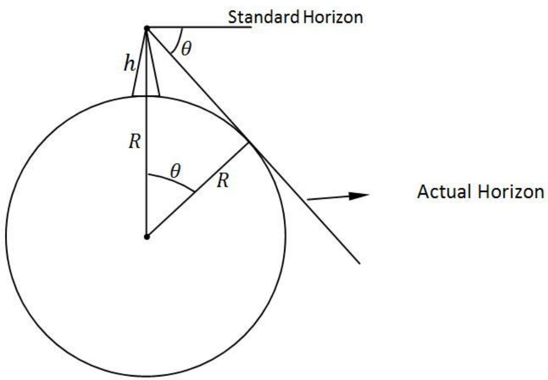
$$\delta_{min} = 90 - Latitude - Refraction - Horizon\ depression$$

(2 points)

$$= 90 - 35^\circ 57' - 34' - 2^\circ 24'$$

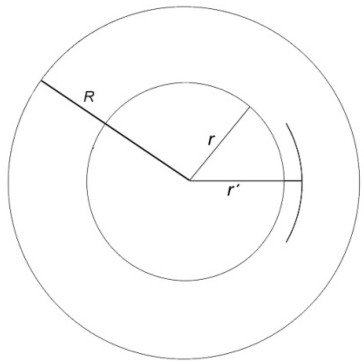
(2 points)

$$\Longrightarrow \delta_{min} = 51^\circ 5'$$



Solution 6:

To solve this problem we must calculate gravitational potential at the center of the cloud, letting $\phi(\infty) = 0$. For a uniform density and spherical mass distribution we have



$$\frac{1}{2}\,m\,v^2(r=R)-\frac{GMm}{R}=E=\frac{1}{2}mv^2(r=0)+\varphi(0)$$ (4 points)

$$v(r=R)=0 \qquad \& \qquad v(r=0)=\sqrt{\frac{GM}{R}}$$ (1 point)

$$\varphi(0)=-\frac{1}{2}mv_0^2-\frac{GmM}{R}$$ (1 point)

$$\varphi(0)=\frac{-1}{2}m\left(\sqrt{\frac{GM}{R}}\right)^2-\frac{GmM}{R}$$ (2 points)

$$\varphi(0) = \frac{-3}{2} \times \frac{GMm}{R}$$

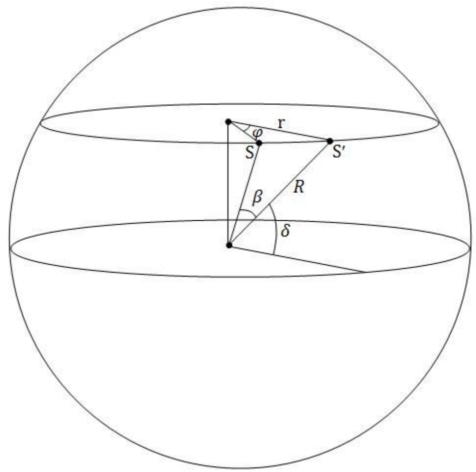
To escape from the cloud, the particle should have total energy equal to zero

$$E = \frac{1}{2}m{v_e}^2 + \phi(r=0) = \frac{1}{2}m{v_e}^2 - \frac{3}{2}\left(\frac{GMm}{R}\right) = 0$$

$${v_e}^2 = \frac{3GM}{R} \quad \rightarrow \quad v_e = \sqrt{\frac{3GM}{R}}$$ (2 points)

Solution 7:

Figure shows that if FOV of telescope is β then we have :



$$\beta = \phi cos\delta$$ (4 points)

As the earth rotate, Vega moves through the FOV with constant angular velocity of the earth

$$\omega = \frac{2\pi}{86164} = 7.29 \times 10^{-5} (rad/s)$$ (2 points)

$$\phi = \omega t = 7.29 \times 10^{-5} \times 5.3 \times 60 = 0.023 \text{ (rad)}$$

$$FOV = \beta = \phi cos\delta = 0.023 cos39^{\circ} = 0.018 \text{ (rad)} \simeq 62\ min$$ (4 points)

Solution 8:

$$\frac{1}{2}Mv_{esc}^2 = \frac{1}{2}M\frac{2GM}{R}$$ (4 points)

The velocity must be smaller than the escape velocity ($v_{esc} \approx \sqrt{2}v_{rms}$) and since the problem is an estimation, any velocities smaller than escape velocity is accepted and therefore the escape velocity can be replaced by v_{rms}.

$$Mv_{rms}^2 = \frac{GM^2}{R}$$ (2 points)

$$M = \frac{Rv_{rms}^2}{G} = \frac{20 \times 3.09 \times 10^{16} \times 9 \times 10^6}{6.67 \times 10^{-11}} = 8.3 \times 10^{34} Kg$$

$$M = 4.2 \times 10^4 M_\odot$$ (4 points)

Solution 9:

In the figure, S is the Sun and R_0 and V_0 are Sun distance and velocity. The distance and velocity of star P is denoted by R and $V = V_0$. The radial velocity of star P respect to the Sun is

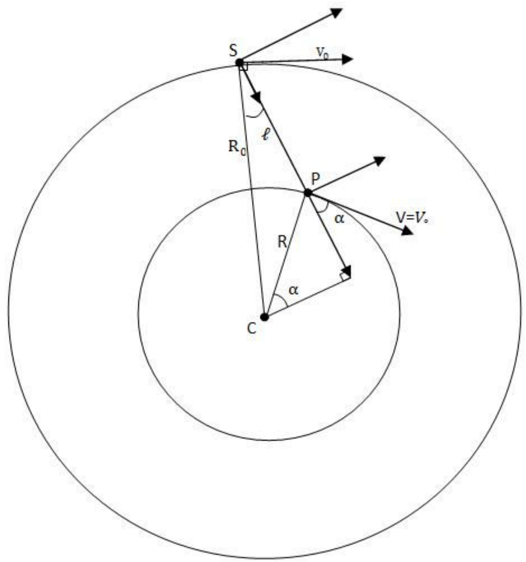
$$V_r = V\cos\alpha - V_0\sin l = V_0(\cos\alpha - \sin l)$$ (4 points)

In SCP triangle we have

$$\frac{\sin l}{R} = \frac{\cos\alpha}{R_0} \Rightarrow \cos\alpha = \frac{R_0}{R}\sin l$$ (1 point)

So

$$V_r = V_0\left(\frac{R_0}{R} - 1\right)\sin l$$ (2 points)



$$\frac{R_0}{R} - 1 = \frac{V_r}{V_0 sinl}$$

$$\frac{R_0}{R} = \frac{V_r + V_0 sinl}{V_0 \sin l}$$

$$R = R_0\frac{V_0 \sin l}{V_r + V_0 sinl}$$ (2 points)

$$R = 8 \times 10^3 \frac{250 sin 15^{\circ}}{100 + 250 sin 15^{\circ}}$$

$$R = 3 \times 10^3\, pc$$ (1 point)

Solution 10:

Because of high conductivity of plasma inside the star, flux of magnetic field will be conserved through contraction then:

$$4\pi R^2 B = 4\pi R_n^2 B_n$$ (6 points)

Where R_n and B_n are radius and magnetic field of neutron star, thus:

$$B_n = \left(\frac{R}{R_n}\right)^2 B = \left(\frac{4 \times 6.96 \times 10^5}{20}\right)^2$$ (2 points)

$$B_n = 1.93 \times 10^{10}\ Gauss$$

$$= 1.93 \times 10^6\ T$$ (2 points)

Solution 11:

In a flat universe

$$\rho = \rho_c = \frac{3H^2}{8\pi G}$$ (4 points)

$$H = 75\ kms^{-1}Mpc^{-1} = 2.4 \times 10^{-18} s^{-1}$$ (2 points)

$$\rho_c = 1.1 \times 10^{-26}\ kgm^{-3}$$ (2 points)

$$n_v = \frac{0.25\ \rho_c}{10^{-5}\ m_e} = 3 \times 10^8\ m^{-3}$$ (2 points)

Solution 12:

The rate of change of solar mass could be estimated from solar luminosity:

(2 points)

$$L_\odot = -\frac{\Delta E}{\Delta t} = -\frac{\Delta M c^2}{\Delta t} = -\dot{M} c^2$$

$$\dot{M} = -\frac{L_\odot}{c^2} = -\frac{3.83 \times 10^{26}}{(3.00 \times 10^8)^2} = -4.26 \times 10^9 (kgs^{-1})$$

Newton second law will give us:

$$\frac{v^2}{r} = \frac{GM}{r^2} \implies v^2 = \frac{GM}{r}$$ (2 points)

where v and r are orbital velocity and orbital radius of the Earth.
From conservation of angular momentum:

$$l = rmv \implies v = \frac{l}{mr}$$ (2 points)

Where m and l are the Earth mass and angular momentum which are constant:

$$\frac{l^2}{m^2 r^2} = \frac{GM}{r} \Rightarrow r = \frac{l^2}{GMm^2} \Rightarrow \dot{r} = -\frac{l^2}{Gm^2}\frac{\dot{M}}{M^2} = -\frac{r^2 m^2 v^2}{GM^2 m^2}\dot{M}$$

$$\Rightarrow \dot{r} = -\frac{(GM/r)r^2}{G}\frac{\dot{M}}{M^2} = -r\frac{\dot{M}}{M} \Rightarrow \frac{\dot{r}}{r} = -\frac{\dot{M}}{M} \Rightarrow \dot{r} = -r\frac{\dot{M}}{M}$$

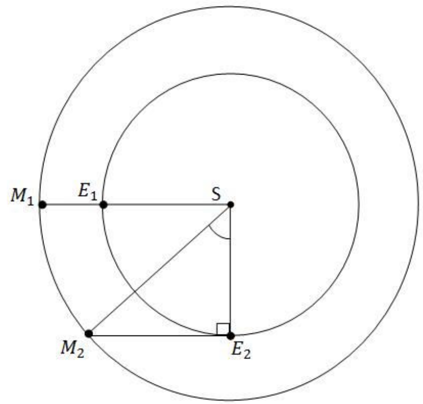
$$\Rightarrow \dot{r} = -\frac{1.50 \times 10^{11} \times 4.26 \times 10^9}{1.99 \times 10^{30}} = 3.21 \times 10^{-10} (m/s)$$

(2 points)

$$\Delta r = 3.19 \times 10^{-10} \times 100 \times 86400 \times 365.24 = 1.01m \quad (for\ 100\ years)$$ (2 points)

Solution 13:

We know the orbital period of Mars then the angle $\angle M_1SM_2$ can be determined simply



$$\angle M_1SM_2 = \frac{106}{687} \times 360 = 55.5°$$ (2 points)

By the same way we determine $\angle E_1SE_2$:

$$\angle E_1SE_2 = \frac{106}{365} \times 360 = 104.5°$$ (2 points)

Then angle $\angle M_2SE_2$ is:

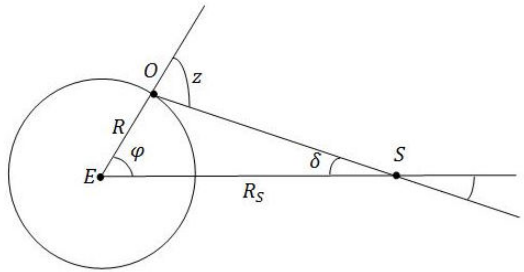
$$\angle M_2SE_2 = 104.5 - 55.5 = 49.0°$$ (2 points)

$$\cos(\angle M_2SE_2) = \frac{SE_2}{SM_2}$$ (2 points)

$$r_{mars} = \frac{SM_2}{SE_2} = \frac{1}{\cos(\angle M_2SE_2)} = \frac{1}{\cos(49°)} = 1.52\ AU$$ (2 points)

Solution 14:

In the figure, the observer is at *O* and the satellite is in *S* the angle $\angle EOS$ will be



$$\angle EOS = 180 - z = 180 - 46.0 = 134°$$ (2 points)

$$\delta = 180 - \varphi - \angle EOS = 10.4°$$ (3 points)

In *EOS* triangle we have:

$$\frac{R_S}{\sin(\angle EOS)} = \frac{R}{\sin\delta}$$ (3 points)

$$\frac{R_S}{R} = \frac{\sin(\angle EOS)}{\sin\delta} = 3.98$$ (2 points)

Solution 15:

Ignoring temperature variation on the stars surface, the brightness of star system will be proportional to projected surface of both stars on plane of the sky. Maximum brightness will occur when two stars are seen like figure 1 and minimum light will happen when one of the stars is in total eclipse and projected surface of the other is a circle with radius *b* (Figure 2). In maximum brightness

$$I_{max} \propto 2\pi ab$$ (3 points)

In minimum brightness

$$I_{min} \propto \pi b^2$$ (3 points)

So

$$\Delta m = -2.5\log\frac{I_{max}}{I_{min}} = -2.5\log\frac{2\pi ab}{\pi b^2} = -2.5\log 4$$ (2 points)

$$\Delta m = -1.5$$ (2 points)

Solution 16:

a) Total energy of the projectile is

$$E = \tfrac{1}{2}mv_\circ^2 - \frac{GMm}{R} = -\frac{GMm}{2R} < 0$$

$E < 0$ means that orbit might be ellipse or circle. As $\theta > 0$, the orbit is an ellipse.
Total energy for an ellipse is

$$E = -\frac{GMm}{2a}$$

Then

$$a = R$$ (7 points)

b) In figure (1) we have

$$OA + O'A = 2a$$

$$O'A = a$$

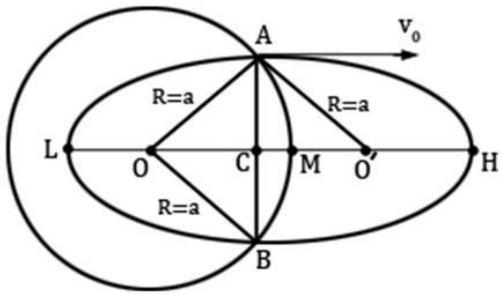


Figure (1)

In OAO' triangle it is obvious that

$$OC = CO'$$

Then C must be the center of the ellipse with the initial velocity vector $v_\circ$ parallel to the ellipse major-axis (LH).

In figure (2)

$$HM = CH - CM = a - (R - R\sin\theta) = R - R + R\sin\theta = R\sin\theta = \frac{R}{2}$$ (15 points)

c) Range of the projectile is $\widehat{AB}$

$$\widehat{AB} = 2\left(\frac{\pi}{2} - \theta\right)R = (\pi - 2\theta)R = \frac{2\pi}{3}R$$

(6 points)

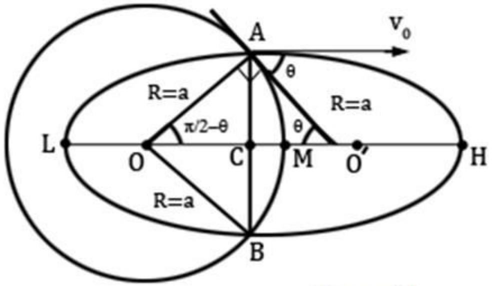


Figure (2)

d) Start with ellipse equation in polar coordinates

$$r = \frac{a(1-e^2)}{1+ecos\varphi}$$

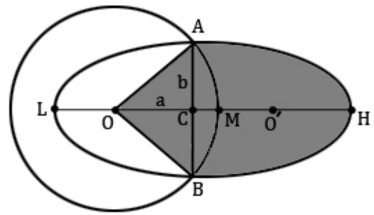
For point A

$$R = \frac{R(1-e^2)}{1-ecos(\frac{\pi}{2}+\theta)}$$

$$e = sin\theta = \frac{1}{2}$$

(5 points)

e) Using Kepler's second law



$$\frac{\Delta S}{S_0} = \frac{\Delta T}{T}$$

$$\Delta S = S_{AOBH} = S_{\Delta AOB} + \frac{S_0}{2}$$

$$= 2 \times \frac{\text{bae}}{2} + \frac{\pi\text{ab}}{2} = \text{bae} + \frac{\pi\text{ab}}{2}$$

$$\frac{\Delta S}{S} = \frac{\text{bae} + \frac{\pi\text{ab}}{2}}{\pi ab} = \frac{e + \frac{\pi}{2}}{\pi} = \frac{0.5 + \frac{\pi}{2}}{\pi}$$

Kepler's third law

$$T = \sqrt{\frac{4\pi^2 R^3}{GM}} = 84.5\, min$$

$$\Delta T = T \times \frac{0.5 + \frac{\pi}{2}}{\pi} = 55.7\, min$$

(12 points)

Solution 17:

a) Relation between the apparent and absolute magnitude is given by

$$m = M + 5log\left(\frac{d}{10}\right)$$ (3 points)

where d is in terms of parsec. Substituting $m = 18$ and $M = -0.2$, results in

(5 points)

$$d = 4.37 \times 10^4\ pc$$

b) Adding the term for the extinction, changes the magnitude distance relation as follows

$$m = M + 0.7x + 5\log(100x)$$

where x is given in terms of kilo parsec. To have a rough value for x, after substituting m and M, this equation reduces to

$$8.2 = 0.7x + 5\log(x)$$ (6 points)

To solve this equation, we examine

$$x = 5, 5.5, 6, 6.5$$

where the best value is obtained roughly $x \cong 6.1\ kpc$. (8 points)

c) For a solid angle Ω, the number of observed red clump stars at the distance in the range of x and $x + \Delta x$ is given by

$$\Delta N = \Omega\, x^2\, n(x) f\, \Delta x$$

So the number of stars observed in Δx is given by

$$\frac{\Delta N}{\Delta x} = \Omega\, x^2\, n(x) f$$

(6 points)

From the relation between the distance and apparent magnitude we have

$$m_1 = M + 5log\left(\frac{x}{10}\right)$$

$$m_2 = M + 5log\left(\frac{x + \Delta x}{10}\right)$$

$$\Delta m = 5log\left(\frac{x + \Delta x}{x}\right)$$

$$\Delta m = 5log\left(1 + \frac{\Delta x}{x}\right)$$

$$\Delta m = \frac{5}{ln10} ln\left(1 + \frac{\Delta x}{x}\right)$$

$$\Delta m = \frac{5}{ln10}\left(\frac{\Delta x}{x}\right)$$

Replacing Δx with Δm, results in

$$\frac{\Delta N}{\Delta m} = \frac{\Delta N}{\Delta x} \times \frac{\Delta x}{\Delta m}$$

So the number of stars for a given magnitude is obtained by (5 points)

$$\frac{\Delta N}{\Delta m} = \frac{\Omega ln10}{5} n(x) x^3 f$$

Finally we substituting x in terms of apparent magnitude using $\mathrm{x} = 10^{\frac{\mathrm{m}+5.2}{5}}$.

In the case of no extinction, we are able to observe the Galaxy beyond the center. So $\frac{dN}{dm}$ has two terms in

$x < R_0$ and $x > R_0$. The relation between x and r for these two cases are

$$x = R_0 - r \qquad x < R_0$$

(6 points)

and

$$x = R_0 + r \qquad x > R_0$$

So in general we can write $\frac{\Delta N}{\Delta m}$ as

(6 points)

$$\frac{\Delta N}{\Delta m} = \frac{\Omega ln10}{5} n_0 \exp\left(\frac{10^{\frac{\mathrm{m}-5.2}{5}}}{R_d}\right) \times 10^{\frac{3(m-5.2)}{5}} f \qquad x < R_0$$

$$\frac{\Delta N}{\Delta m} = \frac{\Omega ln10}{5} \mathrm{n}_0 \exp\left(\frac{2\mathrm{R}_0}{R_d}\right) \exp\left(-\frac{10^{\frac{\mathrm{m}-5.2}{5}}}{R_d}\right) \times 10^{\frac{3(m-5.2)}{5}} f\, \Theta(x_0 - x) \quad x > R_0$$

where $\Theta(x)$ is the step function and x_0 is the maximum observable distance.