Cosmology and Relativity for IOAA

Jakub Vošmera

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Plan for today

1. Relativity
2. Cosmology
3. Problems

Plan for today

1 Relativity

2 Cosmology

3 Problems

Time and space on equal footing: **spacetime**

Special relativity $\implies$ physics in **flat** spacetimes
General relativity $\implies$ physics in **curved** spacetimes

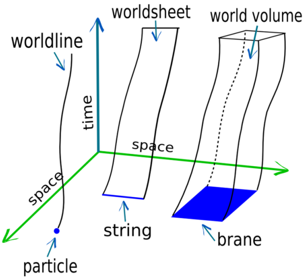
Observer S in spacetime specified by its **reference frame** t, x, y, z

No preferred observers. Speed of light same in all inertial reference frames.

A point in spacetime $[t, x, y, z]$ called an **event**

Time perceived by objects moving in frame S: **proper time** τ

As $[t, x, y, z]$ of moving object change with τ, it traces out a **worldline**



Galileo transformation

Different observer (reference frame) S$'$ moving w.r.t. S with v in the x direction.

Axes of S$'$ labelled by t', x', y', z'. Say S$'$ **boosted** from S.

How are t', x', y', z' and t, x, y, z related?

Up to 1905: **Galileo transformation**

$$t' = t \tag{1a}$$

$$x' = x + vt \tag{1b}$$

$$y' = y \tag{1c}$$

$$z' = z \tag{1d}$$

Velocity $\boldsymbol{u}$ with components u_x, u_y, u_z in S and u'_x, u'_y, u'_z in S$'$ has

$$u'_x = u_x + v \tag{2a}$$

$$u'_y = u_y \tag{2b}$$

$$u'_z = u_z \tag{2c}$$

In particular

$$\boxed{c' = c + v} \tag{3}$$

In 1887 (Michelson & Morley): $c = c' \implies$ Galileo wrong!

Requiring

1. $c' \stackrel{!}{=} c$
2. inverse boost $=$ boost by $-v$

gives **Lorentz transformation**

$$\boxed{\begin{aligned} ct' &= \gamma(ct + \frac{v}{c}x) \\ x' &= \gamma(x + \frac{v}{c}ct) \end{aligned}} \qquad \begin{aligned} (4) \\ (5) \end{aligned}$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{6}$$

We then have

$$u'_x = \frac{\Delta x'}{\Delta t'} = \frac{\frac{\Delta x}{\Delta t} + v}{1 + \frac{v}{c^2}\frac{\Delta x}{\Delta t}} = \frac{u_x + v}{1 + \frac{u_x v}{c^2}} \tag{7}$$

which indeed gives $c' = c$

Proper time, time dilation

$(\Delta t, \Delta x, \Delta y, \Delta z)$ a segment on a world line in S.

Denote

$$\boxed{\Delta\tau^2 = \Delta t^2 - (\Delta x/c)^2 - (\Delta y/c)^2 - (\Delta z/c)^2} \tag{8}$$

Rest frame S_0**:** $\Delta x_0 = \Delta y_0 = \Delta z_0 = 0 \implies \Delta\tau_0 = \Delta t_0$ Boost by v to any S'

$$\Delta\tau'^2 = \frac{\gamma^2}{c^2}(c\Delta t + \frac{v}{c}\Delta x)^2 + \frac{\gamma^2}{c^2}(\Delta x + v\Delta t)^2 + \frac{1}{c^2}\Delta y^2 + \frac{1}{c^2}\Delta z^2 = \ldots = \Delta\tau^2\,. \tag{9}$$

$\implies$ $\Delta\tau$ is **Lorentz invariant**

$\implies$ In any S: $\Delta\tau = \Delta t_0 =$ time interval as perceived in $S_0 \implies \Delta\tau =$ proper time!

$$\Delta\tau = \Delta t\sqrt{1 - \frac{\Delta x/\Delta t}{c^2} - \frac{\Delta y/\Delta t}{c^2} - \frac{\Delta z/\Delta t}{c^2}} = \Delta t/\gamma\,. \tag{10}$$

i.e.

$$\boxed{\Delta t = \gamma\Delta\tau} \tag{11}$$

$\overset{\gamma>1}{\implies}$ **time dilation**

Note that $v = c$ (photons, gravitons) $\implies \Delta\tau = 0$

Relativistic Doppler formula

Two signals emitted in S at $t_1^{\mathrm{e}} = 0$, $t_2^{\mathrm{e}} = T$ and $x_1^{\mathrm{e}} = x_2^{\mathrm{e}} = 0$.

In S$'$ we obtain emission events

$$t_1^{\mathrm{e}\prime} = 0 \tag{12a}$$
$$x_1^{\mathrm{e}\prime} = 0 \tag{12b}$$

and

$$t_2^{\mathrm{e}\prime} = \gamma T \tag{13a}$$
$$x_2^{\mathrm{e}\prime} = \gamma v T \tag{13b}$$

The signals are detected by an observer sitting at $x' = a \gg \gamma v T$ at

$$t_1^{\mathrm{d}\prime} = \frac{a}{c} \tag{14a}$$
$$t_2^{\mathrm{d}\prime} = \frac{a - x_2^{\mathrm{e}\prime}}{c} + t_2^{\mathrm{e}\prime} = \frac{a}{c} + \gamma T - \gamma \frac{v}{c} T\,, \tag{14b}$$

that is $T' = t_2^{\mathrm{d}\prime} - t_1^{\mathrm{d}\prime} = (1 - \frac{v}{c})\gamma T$ satisfies

$$\boxed{T' = T\sqrt{\frac{c-v}{c+v}}} \tag{15}$$

$\Longrightarrow$ **relativistic Doppler effect**

Four-momentum

Energy & momentum depend on reference frame:

$$E' = \gamma(E + \frac{v}{c} c p_x) \tag{16a}$$

$$c p_x' = \gamma(c p_x + \frac{v}{c} E) \tag{16b}$$

with $\varepsilon^2 = E^2 - c^2 p_x^2 - c^2 p_y^2 - c^2 p_z^2$ Lorentz invariant

In S_0 have $p_{x,0} = p_{y,0} = p_{z,0} = 0 \implies \varepsilon = m_0 c^2$ (for all S)

$$\boxed{E^2 = m_0^2 c^4 + p^2 c^2} \tag{17}$$

Massless particles (photons, gravitons) $\implies E = pc$

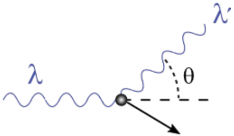
Total four-momentum **conserved** in all dynamical processes:

$$E = \text{const.} \quad p_x = \text{const.} \quad p_y = \text{const.} \quad p_z = \text{const.} \tag{18}$$

Alternative derivation of relativistic Doppler for photons

$$\hbar\omega' = E' = \gamma(E + \frac{v}{c} c \cdot \frac{E}{c}) = \gamma(1 + \frac{v}{c})\hbar\omega \quad \implies \quad \omega' = \sqrt{\frac{c+v}{c-v}}\,\omega\,. \tag{19}$$

Compton scattering



$\Longrightarrow$ What is $\lambda'(\lambda, \theta)$?

Conservation of four-momentum:

$$\frac{hc}{\lambda} + m_\mathrm{e} c^2 = \frac{hc}{\lambda'} + \sqrt{m_\mathrm{e}^2 c^4 + p^2 c^2} \tag{20a}$$

$$\frac{h}{\lambda} = \frac{h}{\lambda'} \cos\theta + p \cos\psi \tag{20b}$$

$$0 = \frac{h}{\lambda'} \sin\theta - p \sin\psi \tag{20c}$$

Therefore

$$p^2 = p^2 \sin^2\psi + p^2 \cos^2\psi = h^2 \left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2} - \frac{2}{\lambda\lambda'} \cos\theta \right) \tag{21}$$

We eventually obtain

$$\boxed{\lambda'(\lambda, \theta) = \lambda + \frac{h}{m_\mathrm{e} c}(1 - \cos\theta)} \tag{22}$$

Light ray propagating in S, not parallel to x

$$E = \hbar\omega \tag{23a}$$
$$cp_x = \hbar\omega\cos\theta \tag{23b}$$
$$cp_y = \hbar\omega\sin\theta \tag{23c}$$

In S$'$ we have

$$E' = \gamma\hbar\omega(1 + \frac{v}{c}\cos\theta) \tag{24a}$$
$$cp'_x = \gamma\hbar\omega(\cos\theta + \frac{v}{c}) \tag{24b}$$

with $cp'_y = cp'_y = \hbar\omega\sin\theta$. Therefore $\tan\theta' = p'_y/p'_x = \sin\theta/\gamma(\cos\theta + \frac{v}{c})$ i.e.

$$\boxed{\cos\theta' = \frac{\cos\theta + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta}} \tag{25}$$

For $\theta = \frac{\pi}{2}$ and $\theta' = \frac{\pi}{2} - \alpha'$ we have $\sin\alpha' = \frac{v}{c} \implies \alpha' \overset{v \ll c}{\approx} \frac{v}{c} \implies$ classical aberration
In astrophysics: relativistic beaming (Problem 2), Poynting–Robertson effect (Problem 3)

Principles of General Relativity

There is no gravitational force

All effects due to gravity can be explained by **curvature** of spacetime.

Proper time interval modified to

$$\Delta\tau^2 = -\frac{1}{c^2}\sum_{\mu,\nu=0}^{3} g_{\mu\nu}\Delta x^\mu \Delta x^\nu \qquad (26)$$

ref. frame coords $x^0 = t, x^1 = x$, $x^2 = y$, $x^3 = z$ and $g_{\mu\nu}(x)$ expressing the curvature

A. Einstein (1915):

1. Spacetime tells matter how to move:

   All objects follow the shortest possible path (geodesic) in spacetime.

2. Matter tells spacetime how to curve:

$$\boxed{R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}}$$

Special relativity formulae continue to hold, but only **locally**.

Schwarzschild spacetime

Vacuum everywhere except for a gravitating body M at $r = 0$ gives proper time interval

$$\boxed{\Delta\tau^2 = \left(1 - \frac{2GM}{rc^2}\right)\Delta t^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{rc^2}\right)^{-1}\Delta r^2} \tag{27}$$

During Δt, observers r_1 and r_2 at rest experience proper time intervals

$$\Delta\tau_i = \left(1 - \frac{2GM}{r_i c^2}\right)^{\frac{1}{2}}\Delta t \quad \Longrightarrow \quad \frac{\Delta\tau_2}{\Delta\tau_1} = \left(\frac{1 - \frac{2GM}{r_2 c^2}}{1 - \frac{2GM}{r_1 c^2}}\right)^{\frac{1}{2}} > 1 \quad \text{for } r_2 > r_1 \tag{28}$$

$\Longrightarrow$ **gravitational time dilation**; note that $\Delta\tau_1/\Delta\tau_1 \to \infty$ as $r_1 \to R_\mathrm{g} = 2GM/c^2$
Consequence: photon emitted at $r = R$ will have following redshift at $r = \infty$

$$\boxed{z = \left(1 - \frac{2GM}{Rc^2}\right)^{-\frac{1}{2}} - 1} \tag{29}$$

$\Longrightarrow$ **gravitational redshift**
Can compute radial photon trajectories by setting $\Delta\tau = 0$

$$\frac{dr}{dt} = \pm\left(1 - \frac{2GM}{rc^2}\right) \tag{30}$$

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Cosmological principle

All present observations indicate that

> On large scales, the universe is homogeneous, isotropic, flat and expanding.

Distance between any two objects can therefore be written as

$$\Delta r(t) = a(t)\,\Delta r \tag{31}$$

where Δr is their present distance and $a(t)$ is increasing in time with $a(t_0) = 1$

Terminology

$\Delta r(t)$	**physical** distance
Δr	**comoving** distance
$a(t)$	scale factor

Corresponding proper time interval (Friedmann-Lemaître-Robertson-Walker)

$$\boxed{\Delta\tau^2 = \Delta t^2 - \frac{1}{c^2}a(t)^2\Delta r^2} \tag{32}$$

with Δr comoving distance.

Evolution of $a(t)$

Flatness condition $\rho = \rho_{\text{crit}}$ gives

$$\boxed{H(t)^2 = \frac{8\pi G}{3}\rho(a)} \tag{33}$$

where $H(t) = \dot{a}(t)/a(t)$ is the **Hubble parameter** (speed of expansion)

Spacetime tells matter how to move:

$$\rho(a) = \begin{cases} \rho_0 a^{-3} & \text{matter} \\ \rho_0 a^{-4} & \text{radiation} \\ \rho_0 & \text{dark energy} \end{cases} \tag{34}$$

Solving (33) for $a(t)$ we have

$$a(t) = \begin{cases} (t/t_0)^{\frac{2}{3}} & \text{matter} \\ (t/t_0)^{\frac{1}{2}} & \text{radiation} \\ e^{H_0(t-t_0)} & \text{dark energy} \end{cases} \tag{35}$$

together with

$$H(t) = \begin{cases} 2/(3t) & \text{matter} \\ 1/(2t) & \text{radiation} \\ H_0 & \text{dark energy} \end{cases} \tag{36}$$

Present-day parameters

Define the present-day density parameters

$$\boxed{\Omega_x = \frac{\rho_{x,0}}{\rho_{\mathrm{crit},0}}} \tag{37}$$

where $x \in \{\mathrm{m}, \mathrm{r}, \Lambda\}$.

Flatness condition (33) can then be rewritten as

$$\boxed{H(a) = H_0 \sqrt{\Omega_{\mathrm{m}} a^{-3} + \Omega_{\mathrm{r}} a^{-4} + \Omega_\Lambda}} \tag{38}$$

Sometimes define the **reduced** Hubble parameter $h = H_0/(100\,\mathrm{km\,s^{-1}\,Mpc^{-1}})$

We then have

$$\rho_{\mathrm{crit},0} = \frac{3H_0^2}{8\pi G} \approx 1.878 \times 10^{-26} h^2\,\mathrm{kg\,m^{-3}} \tag{39}$$

Current ΛCDM fit gives

h	0.6774 ± 0.0046
Ω_{m}	0.3089 ± 0.0062
Ω_Λ	0.6911 ± 0.0062
Ω_{r}	0.0001

Physical lengths expand with $a(t)$

$$\lambda' = \frac{a(t_0)}{a(t)}\lambda\,. \tag{40}$$

for a photon emitted at t detected now. Hence

$$\boxed{a(t) = \frac{1}{1+z}} \tag{41}$$

Important: this is **not** a Doppler redshift. Speeds calculated using Doppler formula have no physical basis.

Can use (41) to rewrite (38) as

$$\boxed{H(z) = H_0\sqrt{\Omega_\mathrm{m}(1+z)^3 + \Omega_\mathrm{r}(1+z)^4 + \Omega_\Lambda}} \tag{42}$$

Some important redshifts

Matter-radiation eq.	3000
Recombination	1100
Dark ages	$1100 - 20$
Reionization	$20 - 6$
GN-z11	11.09

We can measure intensity I of a source with luminosity L. Ignoring extinction have

$$I = \frac{L}{4\pi r_{\mathrm{L}}^2}\,, \tag{43}$$

where r_{L} defines the **luminosity distance**.

To relate this to the comoving distance r, note that

1. energies of all photons decrease by $(1+z)^{-1}$
2. time intervals between photons increase by $(1+z)$

This gives

$$\boxed{r_{\mathrm{L}} = r(1+z)} \tag{44}$$

We can also measure the angular diameter δ of a source with physical diameter D. We have

$$\delta = \frac{D}{r_{\mathrm{A}}} \tag{45}$$

Converting D to comoving scale, we obtain

$$\boxed{r_{\mathrm{A}} = \frac{r}{1+z}} \tag{46}$$

Basic trick:

Comoving dist. r_{AB} between A and B can be found by connecting them with a photon.

Comoving distance Δr traversed by a photon over interval Δt

$$\Delta r = \frac{\Delta r(t)}{a(t)} = \frac{c\Delta t}{a(t)} \tag{47}$$

Hence (can be also derived from the FLRW proper time interval (32))

$$\boxed{r_{\mathrm{AB}} = c\int_{t_{\mathrm{A}}}^{t_{\mathrm{B}}} \frac{dt}{a(t)}\,.} \tag{48}$$

This is particularly useful: all things we see on the sky are connected to us by a photon!

Use (49), (41) and (42) to calculate comoving dist. r at redshift z as (**Hubble's law**)

$$\boxed{r(z) = c\int_0^z \frac{dz'}{H(z)} \overset{z\ll 1}{\approx} \frac{cz}{H_0}\,.} \tag{49}$$

Note that the approximation becomes exact in Λ-dominated universe $H(z) = H_0 =$ cons

Derivation of Hubble's law**

We have

$$r(z) = c\int_t^{t_0} \frac{dt}{a} = c\int_{a(z)}^{1} \frac{da}{a\dot{a}} = c\int_{a(z)}^{1} \frac{da}{a^2}\frac{1}{H(a)} \tag{50}$$

where in the last step we used the definition $H = \dot{a}/a$.

Further changing the variables to redshift, we obtain

$$r(z) = c\int_0^z \frac{dz'}{H(z')}\underbrace{\frac{1}{a(z')^2}\frac{1}{(1+z')^2}}_{=1} = c\int_0^z \frac{dz'}{H(z')}\,. \tag{51}$$

Hence, for $z \ll 1$, it follows that

$$r(z) \approx \frac{cz}{H_0}\,. \tag{52}$$

Some omnipresent derivatives and integrals

Derivatives:

$$\frac{d}{dt} t^p = p t^{p-1} \tag{53a}$$

$$\frac{d}{dt} e^{Ht} = H e^{Ht} \tag{53b}$$

$$\frac{d}{dz} (1+z)^p = p(1+z)^{p-1} \tag{53c}$$

Integrals:

$$\int dt\, t^p = \frac{t^{p+1}}{p+1} \quad p \neq -1 \tag{54a}$$

$$\int dt\, e^{Ht} = \frac{1}{H} e^{Ht} \tag{54b}$$

$$\int dz\, (1+z)^p = \frac{z^{p+1}}{p+1} \quad p \neq -1 \tag{54c}$$

matter-dominated universe ≡ Einstein – de Sitter universe

Have $a(t) = (t/t_0)^{\frac{2}{3}}$, $H(z) = H_0(1+z)^{\frac{3}{2}}$ and so

$$r(z) = \frac{c}{H_0}\int_0^z dz'\,(1+z')^{-\frac{3}{2}} = \frac{2c}{H_0}[1-(1+z)^{-\frac{1}{2}}] \tag{55a}$$

$r_{\mathrm{L}}(z)$ and $r_{\mathrm{A}}(z)$ follow straightforwardly. For $z \ll 1$, we indeed have

$$\frac{2c}{H_0}[1-(1+z)^{-\frac{1}{2}}] \approx \frac{2c}{H_0}[1-(1-\frac{1}{2}z)] = \frac{cz}{H_0} \tag{56}$$

Λ-dominated universe ≡ de Sitter universe

Have $a(t) = e^{H_0(t-t_0)}$, $H(z) = H_0 = \text{const.}$ and so

$$r(z) = \frac{c}{H_0}\int_0^z dz' = \frac{cz}{H_0} \tag{57a}$$

$\Longrightarrow$ Hubble's law linear for all z

Cosmological horizons

Past comoving horizon: maximum comoving distance from which the particles could have travelled to influence us at present

$$\boxed{r_{\mathrm{PCH}} = c\int_0^{t_0} \frac{dt}{a(t)} = c\int_0^{\infty} \frac{dz}{H(z)}} \tag{58}$$

Future comoving horizon: maximum comoving distance to which we can travel starting at present.

$$\boxed{r_{\mathrm{FCH}} = c\int_{t_0}^{\infty} \frac{dt}{a(t)} = c\int_{-1}^{0} \frac{dz}{H(z)}} \tag{59}$$

Not all universes have past and future horizons, i.e. it can happen that $r_{\mathrm{PCH,FCH}} = \infty$

Examples: in Einstein – de Sitter have

$$r_{\mathrm{PCH}} = \frac{c}{H_0}\int_0^{\infty} dz\,(1+z)^{-\frac{3}{2}} = \frac{2c}{H_0}\,, \quad r_{\mathrm{FCH}} = \frac{c}{H_0}\int_{-1}^{0} dz\,(1+z)^{-\frac{3}{2}} = \infty \tag{60}$$

while in de Sitter universe have

$$r_{\mathrm{PCH}} = \frac{c}{H_0}\int_0^{\infty} dz = \infty\,, \quad r_{\mathrm{FCH}} = \frac{c}{H_0}\int_{-1}^{0} dz = \frac{c}{H_0} \tag{61}$$

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Problem 1: Neutron star

Observed spectrum of a galactic neutron star contains a line at $E = 400\,\mathrm{keV}$. This line should correspond to the $e^+e^- \to 2\gamma$ annihilation. Assume $m_\mathrm{e} = 511\,\mathrm{keV}/c^2$

a) Calculate the radius R of the neutron star as a multiple of $R_\mathrm{g} = 2GM/c^2$.

b) Assuming that the bulk of the star consist of nuclear matter with density $\rho = 4 \times 10^{17}\,\mathrm{kg\,m^{-3}}$, find $M/M_\odot$ and R in km.

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Solution:

a) Annihilation occurs preferentially when kinetic energies are much lower than rest masses. Hence $E_\gamma = m_\mathrm{e} c^2 = 511\,\mathrm{keV}$ and so $z = E_\gamma/E - 1 \approx 0.28$. This gives $R/R_\mathrm{g} \approx 2.6$.

b) We obtain

$$2.6 \approx \frac{R}{R_\mathrm{g}} = \frac{Rc^2}{2GM} = \frac{3c^2}{8\pi\rho G R^2} \implies R \approx 12.5\,\mathrm{km}$$

and $M = \frac{4}{3}\pi\rho R^3 \approx 1.6 M_\odot$.

Problem 2: GN-z11

Assume that $H_0 = 67.8\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$

a) Find the value of a^* and z^* when matter and dark energy had equal densities.

Consider the high-redshift galaxy GN-z11 with $z = 11.09$.

b) Find the comoving distance to GN-z11.

c) What was the age of the universe when the photons, which we observe today, were emitted from GN-z11? What is the actual distance travelled by these photons?

d) Given its observed angular diameter $\delta = 0.5''$, find its physical diameter D.

e) Given its magnitude $m = 26.8\,\mathrm{mag}$, estimate its luminosity (neglecting extinction).

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Solution:

a) We need $\Omega_{\mathrm{m},0}(1+z)^3 = \Omega_{\Lambda,0}$, i.e. $a^* = (\Omega_{\mathrm{m},0}/\Omega_{\Lambda_0})^{\frac{1}{3}} \approx 0.75$ and $z^* = 1/a - 1 \approx 0.33$

b) We have

$$r = \int_0^z \frac{cdz'}{H(z')} = \frac{c}{H_0}\int_0^{z^*} dz' + \frac{c}{H_0}\int_{z^*}^{z} dz'(1+z')^{-\frac{3}{2}} = \frac{2c}{H_0}[\frac{z^*}{2} + (1+z^*)^{-\frac{1}{2}} - (1+z)^{-\frac{1}{2}}]$$

i.e. $r = 6.6\,\mathrm{Gpc}$. Linearised Hubble's law would instead give 49 Gpc.

c) Have $(t^*/t)^{2/3} = (1+z)/(1+z^*)$ and $a^* = e^{H_0(-t_0+t^*)}$, so $t^* \approx 9.65 \times 10^9$ y and so $t \approx 350 \times 10^6$ y. The photons therefore travelled distance $\approx 13.4 \times 10^9$ ly ≈ 4.1 Gpc.

d) The angular diameter distance is equal to $r_\mathrm{A} = r/(1+z) \approx 0.55$ Gpc. The linear size then follows as $D = r_\mathrm{A}\delta \approx 1200$ pc.

e) We have luminosity distance $r_\mathrm{L} = (1+z)r \approx 80$ Gpc. We then have

$$m - M_\odot = -2.5 \log \frac{(10\,\mathrm{pc})^2 L}{r_\mathrm{L}^2 L_\odot} \implies L = L_\odot \left(\frac{r_\mathrm{L}}{10\,\mathrm{pc}}\right)^2 10^{-0.4(m-M_\odot)}$$

i.e. $L \approx 10^{11} L_\odot$.

Problem 3: The further the bigger

Assuming matter dominated universe and a galaxy of fixed physical size D, show that there is a value z^* of redshift where the observed angular diameter of the galaxy starts increasing with increasing z.

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Solution: we have

$$\delta(z) = \frac{D}{r_{\mathrm{A}}(z)} = \frac{1+z}{r(z)}\,D\,,$$

where

$$r(z) = c\int_0^z \frac{dz'}{H(z')} = \frac{c}{H_0}\int_0^z (1+z)^{-\frac{3}{2}} = \frac{2c}{H_0}[1-(1+z)^{-\frac{1}{2}}]\,,$$

so

$$\delta(z) = \frac{H_0 D}{2c}\,\frac{1+z}{1-(1+z)^{-\frac{1}{2}}}\,.$$

This can be shown to have a minimum at $z^* = 5/4$.

Problem 4: Gershtein – Zeldovich bound

We have $n_{\nu_\mathrm{e}} = n_{\nu_\mu} = \ldots = (3/11)n_\gamma$, where n_γ and $n_{\nu_\mathrm{e}}, n_{\nu_\mu}, \ldots$ are the number densities of relic photons and relic neutrinos, respectively, and today we have $n_\gamma \doteq 410\,\mathrm{cm}^{-3}$. Assuming that relic neutrinos are the sole component of dark matter in our universe, find the sum $m_0 = m_{\nu_\mathrm{e},0} + m_{\nu_\mu,0} + \ldots$ of the rest masses of all neutrino species. Write your answer in eV/c^2.

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Solution: we have

Current density of relic neutrinos:

$$\rho_\nu = \rho_{\nu_e} + \rho_{\nu_\mu} + \ldots = m_{\nu_e,0}n_{\nu_e} + m_{\nu_\mu,0}n_{\nu_\mu} + \ldots = \frac{3}{11}n_\gamma m_0\,,$$

(entropy conservation for $e^+e^- \to 2\gamma$)

The universe is flat $\implies$ $\rho_c = 3H_0^2/(8\pi G)$.

Dark matter consists of neutrinos only $\implies$ $\rho_\nu = \rho_{\mathrm{DM}}$

$$\rho_\nu = \frac{3}{11}n_\gamma m_0 = X\rho_c = \frac{3XH_0^2}{8\pi G} \implies m_0 = \frac{11XH_0^2}{8\pi G n_\gamma} \doteq 11\,\mathrm{eV}/c^2\,.$$

This gives an upper bound (Gershtein–Zeldovich)

Problem 5: Relativistic beaming effect

Consider a star with magnitude $m = 0\,\mathrm{mag}$ and zero radial speed. What would be the magnitude if we started moving towards the star at speed $v = 0.5c$?

Problem 6: Breakthrough starshot*

How long would it take to accelerate a solar sail with $m = 1\,\mathrm{g}$ to $v = 0.2c$ with a 100 GW laser?

Problem 7: Sunyaev – Zeldovich effect

Consider electron at rest and an incident photon, wavelength λ.

a) Fill in the gaps in the derivation of the Compton formula for $\lambda'(\lambda, \theta)$

Now consider a galaxy cluster with total mass $M = 1.2 \cdot 10^{15} M_{\mathrm{S}}$.

b) Consider intracluster gas at equilibrium with particle mass m_0. Derive an approximate formula for its typical temperature.

c) Evaluate this temperature for hydrogen and free electron gas. Comment.

Interaction of intracluster electron gas with CMB photons gives rise to the **Sunyaev – Zel'dovich effect**. The cluster is at rest relative to CMB frame.

d) Find the typical magnitude of change of wavelength of a single CMB photon scattered by an intracluster electron.

You can simplify your analysis by considering only co-linear collisions in the CMB frame.

e) Consider the net change of wavelength of CMB photons and decide whether it is positive (i.e. the photons get cooler) or negative (the photons are heated).

f) Estimate the corresponding change in the CMB temperature.