Theoretical problems

1. (S) Calculate the diameter of a telescope in which the star ζ CMi (5.13^m) would appear as bright as Sirius (-1.46^m) to the unaided eye. Assume that the diameter of the pupil of the eye is 6 mm and that 30 % of the light incident on the telescope collecting area is lost. **(30 p)**

2. (S) Compute the rotational period and magnetic field of a main sequence (MS) star with 1 solar mass, assuming that it changes its size according the following table (it became giant or white dwarf). Assume that during the stellar evolution the stars conserve their angular momentum as well as magnetic induction flux. **(30 p)**

	radius	rot. period	magnetic field
MS star	700 000 km	25 days	10^{-4} T
Giant	200 $R_\odot$		
White Dwarf	20 000 km		

3. (S) A solar-sail spaceship (with its sail folded) orbits the Sun around a circular orbit of radius a. The reflectance of the sail is k and its total area is S. The mass and luminosity of the Sun are denoted by M and L. At one moment, the ship hoists its sails towards directly to the Sun. Find the new semi-major axis of the ship after it hoists its sail such that it always faces the Sun. What condition on k must be satisfied so that the ship does not leave solar system? Give your answer in terms of a, m, S, M, L and fundamental constants. **(40 p)**

4. (M) The small radius and high density of a white dwarf can be traced to the behavior of a degenerate electron gas in a gravitational field. Chandrasekhar found that the more massive the white dwarf, the *smaller* its radius. The equation of state for an ordinary gas is very simple: $p = nkT$ or $p = k\varrho T/m$, where p is the pressure, n the number density, ϱ the mass density, m the average mass of one particle, T the temperature, and k Boltzmann's constant. If the density is high enough to render the gas degenerate (e.g. in white dwarfs), a different equation of state applies: $p = K\varrho^{5/3}$, where K is a constant. Derive an expression between the radius R and the mass M of a white dwarf. *Hint:* You may need an approximate expression between the internal pressure, which balances the gravity, and the mass and radius of the star. To derive this equation divide the sphere of the star by a plane into two equal halves. **(50 p)**

5. (M) In this problem, let us first assume that the universe was radiation-dominated from the Big Bang up to recombination $z_\mathrm{CMB} \sim 1100$ and has been matter-dominated ever since. You can also assume that the estimated age of the universe is $t_0 = 13.80 \times 10^9$ yr, the Hubble parameter is $H_0 = 67.8\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and the current energy of CMB is about 10^{-3} eV.

a) Find the angular distance of two points on the CMB map, which could have exchanged a photon since the Big Bang. (35 p)

You should find that this distance is less than 180°. This gives a paradox (called horizon paradox) because even the antipodal points of CMB are well correlated. Assume therefore, that the radiation-dominated period was preceded by a very short period of rapid exponential expansion called inflation. The standard model requires the inflation to end when typical energies in the universe reach the GUT scale 10^{15} GeV.

b) How many times (N) the universe would have had to expand during inflation in order for the horizon paradox to disappear? Express your answer as $N = e^p$ for some number p. (35 p)

(70 p)

6. (L) In this question you will explore some possibilities of determining the position of a gravitational wave (GW) signal on the sky based on timing data from the three running detectors: LIGO at Hanford & Livingston and Virgo near Pisa. For the purposes of this question, let us assume that the gravitational waves propagate at the speed of light.

The three GW detectors are located at geographical coordinates which we denote by ϕ_i, λ_i (latitude, longitude), where $i = \mathrm{H, L, V}$ stand for Hanford, Livingston and Virgo. At this point we leave the values of these coordinates unspecified. Let us denote the radius of the Earth by R.

a) Express the distance d_{ij} between detectors i and j in terms of ϕ_i, λ_i, ϕ_j, λ_j and R. By distance we mean the length of the straight line segment connecting the detectors (which goes under the terrestrial surface). (15 p)

Denote by Δt_{ij} the detection time difference for the detectors i and j. The value of Δt_{ij} will typically be of the order of milliseconds. The knowledge of Δt_{ij} for a single pair of detectors will restrict the locus of possible positions of the GW source on the sky to a circle. Note that in general this will *not* be a great circle.

Initially, let us assume that the detection event occurred at 00:00 of Greenwich sidereal time.

b) Given the knowledge of Δt_{ij} for a single pair of detectors i, j, determine the circle on which the possible positions of the GW source lie. You should answer this question by specifying the equatorial coordinates (right-ascension and declination) of the centre of this circle and its angular radius. Express your answer in terms of Δt_{ij}, ϕ_i, λ_i, ϕ_j, λ_j and R. (20 p)

Let us now assume that the detection occurred at some general Greenwich sidereal time Θ.

c) Redo part b) assuming general Greenwich sidereal time Θ of the detection. Express your answer in terms of Θ, Δt_{ij}, ϕ_i, λ_i, ϕ_j, λ_j and R. (5 p)

We will now consider a particular detection event, namely GW170817, which is widely assumed to be due to a binary neutron star merger. The signal arrived to the Earth at 12:41 GMT on August 17, 2017. It arrived first at Virgo, then 22 milliseconds later at LIGO-Livingston and another 3 milliseconds later at LIGO-Hanford (see arXiv:1710.05833). The geographic coordinates of the three detectors are as follows

$$
\begin{aligned}
\phi_\mathrm{H} &= 46.4552^\circ\,\mathrm{N} \qquad & \lambda_\mathrm{H} &= 119.4075^\circ\,\mathrm{W}\,, \\
\phi_\mathrm{L} &= 30.5021^\circ\,\mathrm{N} \qquad & \lambda_\mathrm{L} &= 90.7479^\circ\,\mathrm{W}\,, \\
\phi_\mathrm{V} &= 43.6313^\circ\,\mathrm{N} \qquad & \lambda_\mathrm{V} &= 10.5045^\circ\,\mathrm{E}\,.
\end{aligned}
$$

d) Calculate the value of Θ which corresponds to the detection of GW170817. The equation of time and the R.A. of the Sun for 12:00 GMT of the given date are -4 min and $9^\mathrm{h}48^\mathrm{m}$, respectively. (10 p)

e) Using your results from b) and c), calculate the centres and angular radii of the circles on the sky corresponding to the pairs VH, VL and LH. (10 p)

f) Does the measurement of detection time differences in a system of three detectors determine the position of the source on the sky uniquely? Calculate the possible positions (right-ascension and declination) of the GW170817 source on the sky, which are consistent with the measured detection time differences. Compare your results with the coordinates $\alpha_\mathrm{opt} = 13^\mathrm{h}9^\mathrm{m}48^\mathrm{s}$, $\delta_\mathrm{opt} = -23^\circ 22' 53''$ of the optical counterpart. *Hint:* Wherever justified, you may replace equation of a circle on a sphere by the equation of a circle on a plane with the same centre and radius. (20 p)

(80 p)

PROBLEM No. 1

(S) Calculate the diameter of a telescope in which the star ζ CMi (5.13^{m}) would appear as bright as Sirius (-1.46^{m}) to the unaided eye. Assume that the diameter of the pupil of the eye is 6 mm and that 30 % of the light incident on the telescope collecting area is lost. (**30 p**)

SOLUTION:

Denote the apparent brightness and flux of the stars by m_1, F_1 (ζ CMi) and m_2, F_2 (Sirius), then from the definition of the magnitude scale we get

$$m_2 - m_1 = -2.5 \log \frac{AF_2}{F_1} \qquad (6\,\mathrm{p})$$

where A is the reflectance of the telescope, in our case $A = 0.7$.

From the equation above we can derive the ratio of the fluxes as

$$\frac{F_2}{F_1} = \frac{1}{A} \times 10^{-0.4(m_2 - m_1)}. \qquad (6\,\mathrm{p})$$

The ratio of the fluxes is equal to the square of the ratio of the telescope diameters, D_{tel} and D_{pup}:

$$\left(\frac{D_{\mathrm{tel}}}{D_{\mathrm{pup}}}\right)^2 = \frac{1}{A} \times 10^{-0.4(m_2 - m_1)}, \qquad (6\,\mathrm{p})$$

so

$$D_{\mathrm{tel}} = D_{\mathrm{pup}} \sqrt{\frac{1}{A} \times 10^{-0.4(m_2 - m_1)}}. \qquad (6\,\mathrm{p})$$

$$D_{\mathrm{tel}} = 149\,\mathrm{mm} \qquad (6\,\mathrm{p})$$

PROBLEM No. 2

(S) Compute the rotational period and magnetic field of a main sequence (MS) star with 1 solar mass, assuming that it changes its size according the following table (it became giant or white dwarf). Assume that during the stellar evolution the stars conserve their angular momentum as well as magnetic induction flux. (**30 p**)

	radius	rot. period	magnetic field
MS star	700 000 km	25 days	10^{-4} T
Giant	200 $R_\odot$		
White Dwarf	20 000 km		

SOLUTION:

From the law of conservation of angular momentum we can derive the period:

$$J\omega = \text{const} \rightarrow \frac{2}{5}mR^2 = \text{const} \rightarrow \frac{R^2}{T} = \text{const} \qquad (7\,\text{p})$$

The ratio of square of size and the period during the stellar evolution is constant. We can assume that magnetic induction flux is constant too.

$$BS = \text{const} \rightarrow B \times 4\pi R^2 = \text{const} \rightarrow BR^2 = \text{const} \qquad (7\,\text{p})$$

	radius	rot. period	magnetic field
MS star	700 000 km	25 days	10^{-4} T
Giant	200 $R_\odot$	2700 years	2.5×10^{-9} T
White Dwarf	20 000 km	29 min	0.123 T

(16 p)

PROBLEM No. 3

(S) A solar-sail spaceship (with its sail folded) orbits the Sun around a circular orbit of radius a. The reflectance of the sail is k and its total area is S. The mass and luminosity of the Sun are denoted by M and L. At one moment, the ship hoists its sails towards directly to the Sun. Find the new semi-major axis of the ship after it hoists its sail such that it always faces the Sun. What condition on k must be satisfied so that the ship does not leave solar system? Give your answer in terms of a, m, S, M, L and fundamental constants. (**40 p**)

SOLUTION:

The force acting on the sail is equal to the rate of change of momentum of the ship

$$F_{\text{rad}} = \frac{\Delta p}{\Delta t} = (1+k)\frac{LS}{4\pi c r^2}\,. \qquad (5\,\text{p})$$

The resulting force acting on the ship towards the Sun is

$$F = F_{\text{g}} - F_{\text{rad}} = \left[1 - \frac{(1+k)LS}{4\pi GMmc}\right]\frac{GMm}{r^2} \equiv \frac{G'Mm}{r^2} \qquad (5\,\text{p})$$

where we define the effective gravitational constant

$$G' = \left[1 - \frac{(1+k)LS}{4\pi GMmc}\right] G. \qquad (6\,\text{p})$$

Before the ship hoists its sail, its orbital speed is

$$v = \sqrt{\frac{GM}{a}}\,. \qquad (6\,\text{p})$$

Conservation of mechanical energy for the situation after the ship unfolds the sail gives

$$\frac{1}{2}mv^2 - \frac{G'Mm}{a} = -\frac{G'Mm}{2a'}\,, \qquad (6\,\text{p})$$

that is

$$a' = \frac{G'}{2G' - G}\,a = \frac{4\pi GMmc - (1+k)LS}{2\pi GMmc - (1+k)LS}\,a\,. \qquad (6\,\text{p})$$

In order for the ship to remain within the solar system, we need $G' > G/2$, which gives

$$k + 1 < \frac{2\pi GMmc}{LS}\,. \qquad (6\,\text{p})$$

PROBLEM No. 4

(M) The small radius and high density of a white dwarf can be traced to the behavior of a degenerate electron gas in a gravitational field. Chandrasekhar found that the more massive the white dwarf, the *smaller* its radius. The equation of state for an ordinary gas is very simple: $p = nkT$ or $p = k\varrho T/m$, where p is the pressure, n the number density, ϱ the mass density, m the average mass of one particle, T the temperature, and k Boltzmann's constant. If the density is high enough to render the gas degenerate (e.g. in white dwarfs), a different equation of state applies: $p = K\varrho^{5/3}$, where K is a constant. Derive an expression between the radius R and the mass M of a white dwarf. *Hint:* You may need an approximate expression between the internal pressure, which balances the gravity, and the mass and radius of the star. To derive this equation divide the sphere of the star by a plane into two equal halves. (**50 p**)

SOLUTION:

How much pressure is needed to support a star? We can estimate this roughly as follows.

Imagine the star divided by a plane down the middle. The two halves each have a mass of $M/2$, where M is the mass of the star, and the centers of these two hemisphere are approximately a distance R apart, where R is the radius of the star. The force pulling the two halves together is, according to Newton's law of gravitation:

$$F = G\,\frac{\frac{M}{2}\,\frac{M}{2}}{R^2} = \frac{GM^2}{4R^2} \qquad (10\,\mathrm{p})$$

The force keeping the two halves apart is the pressure, p, times the area of the two hemispheres, πR^2.

For the gravitational force to equal the pressure force, we must have

$$p\left(\pi R^2\right) = \frac{GM^2}{4R^2}, \text{ or } p = \frac{GM^2}{4\pi R^4}. \qquad (10\,\mathrm{p})$$

Now we have two expressions for the pressure. Putting these two relations together, we have

$$p = \frac{GM^2}{4\pi R^4} = K\varrho^{5/3}. \qquad (5\,\mathrm{p})$$

If we express the density ϱ in terms of the mass and radius as

$$\varrho = \frac{M}{\frac{4}{3}\pi R^3}, \qquad (5\,\mathrm{p})$$

we can write

$$\frac{GM^2}{4\pi R^4} = K\left(\frac{M}{\frac{4}{3}\pi R^3}\right)^{5/3} = \frac{K}{\left(\frac{4}{3}\pi\right)^{5/3}}\,\frac{M^{5/3}}{R^5}. \qquad (10\,\mathrm{p})$$

Solving for R in terms of M gives:

$$R = \frac{4\pi K}{G\left(\frac{4}{3}\pi\right)^{5/3}}\,\frac{1}{M^{1/3}} \qquad (10\,\mathrm{p})$$

That is, the radius of a degenerate star is proportional to the inverse cube root of the mass.

PROBLEM No. 5

(M) In this problem, let us first assume that the universe was radiation-dominated from the Big Bang up to recombination $z_{\mathrm{CMB}} \sim 1100$ and has been matter-dominated ever since. You can also assume that the estimated age of the universe is $t_0 = 13.80 \times 10^9\,\mathrm{yr}$, the Hubble parameter is $H_0 = 67.8\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and the current energy of CMB is about $10^{-3}\,\mathrm{eV}$.

a) Find the angular distance of two points on the CMB map, which could have exchanged a photon since the Big Bang. (35 p)

You should find that this distance is less than 180°. This gives a paradox (called horizon paradox) because even the antipodal points of CMB are well correlated. Assume therefore, that the radiation-dominated period was preceded by a very short period of rapid exponential expansion called inflation. The standard model requires the inflation to end when typical energies in the universe reach the GUT scale $10^{15}\,\mathrm{GeV}$.

b) How many times (N) the universe would have had to expand during inflation in order for the horizon paradox to disappear? Express your answer as $N = e^p$ for some number p. (35 p)

(70 p)

SOLUTION:

a)

1. **Approximative solution without calculus:** Denote by t_0 the present age of the universe. Let us first calculate at which time t_{CMB} after the big bang the CMB was radiated. Since the universe was matter dominated in the period between recombination and the present epoch, We have $(t_{\mathrm{CMB}}/t_0)^{2/3} = 1/(1+z_{\mathrm{CMB}})$. Consider points A and B. At the Big bang ($t = 0$), a photon was emitted from A towards B, where it was received at t_{CMB}. The co-moving distance the photon travelled was approximately
$$d_{\mathrm{AB}} \approx \frac{ct_{\mathrm{CMB}}}{a_{\mathrm{CMB}}} = ct_0(1+z_{\mathrm{CMB}})^{-1/2}\,. \qquad (13\,\mathrm{p})$$
The co-moving distance which photons travelled from recombination to present is approximately
$$d_{\mathrm{CMB}} \approx c(t_0 - t_{\mathrm{CMB}}) \approx ct_0\,. \qquad (13\,\mathrm{p})$$
Hence
$$\theta_{\mathrm{AB}} = d_{\mathrm{AB}}/d_{\mathrm{CMB}} \approx (1+z_{\mathrm{CMB}})^{-1/2} \approx 1.7^\circ\,. \qquad (9\,\mathrm{p})$$

2. **Exact solution:** Denote by t_0 the present age of the universe. Let us first calculate at which time t_{CMB} after the big bang the CMB was radiated. Since the universe was matter dominated in the period between recombination and the present epoch, we have $(t_{\mathrm{CMB}}/t_0)^{2/3} = 1/(1+z_{\mathrm{CMB}})$. Consider now points A and B. At the Big bang ($t = 0$), a photon was emitted from A towards B, where it was received at t_{CMB}. The co-moving distance traversed by the photon on the course from A to B is therefore
$$d_{\mathrm{AB}} = c\int_0^{t_{\mathrm{CMB}}} \frac{dt}{a(t)} = 2ct_0(1+z_{\mathrm{CMB}})^{-1/2}\,. \qquad (13\,\mathrm{p})$$
The co-moving distance between an observed at the Earth and the surface of last scattering can be computed as
$$d_{\mathrm{CMB}} = c\int_{t_{\mathrm{CMB}}}^{t_0} \frac{dt}{a(t)} = 3ct_0[1-(1+z_{\mathrm{CMB}})^{-1/2}] \approx 3ct_0\,. \qquad (13\,\mathrm{p})$$
The angular distance between A and B as observed from the Earth is therefore
$$\theta_{\mathrm{AB}} = \frac{d_{\mathrm{AB}}}{d_{\mathrm{CMB}}} \approx \frac{2}{3}(1+z_{\mathrm{CMB}})^{-1/2} \approx 1.15^\circ\,. \qquad (9\,\mathrm{p})$$

b) In order for the paradox to disappear, we need $\theta_{\mathrm{AB}} = \pi$, that is we need the new co-moving distance d'_{AB} traversed by a photon emitted by A at the Big Bang and detected by B at t_{CMB} to satisfy $d'_{\mathrm{AB}} = 2d_{\mathrm{CMB}}$. Let us denote by t_{start} and t_{end} the times when the inflationary period started and ended. During the inflationary period, we have $a(t) \propto e^{Ht}$, where the Hubble parameter $H = H_{\mathrm{start}} = H_{\mathrm{end}}$ is constant during the inflationary period. We can therefore write

$$2d_{\mathrm{CMB}} = d'_{\mathrm{AB}} \approx \frac{c}{H}(z_{\mathrm{start}} - z_{\mathrm{end}}) \approx \frac{cz_{\mathrm{start}}}{H} \approx \frac{cN}{a_{\mathrm{end}}H}\,. \qquad (12\,\mathrm{p})$$

That is

$$N = 6t_0H_0\frac{a_{\mathrm{end}}H_{\mathrm{end}}}{a_0H_0}\,, \qquad (12\,\mathrm{p})$$

where

$$\frac{a_{\mathrm{end}}H_{\mathrm{end}}}{a_0H_0} = \frac{a_{\mathrm{end}}}{a_0}\left(\frac{a_{\mathrm{end}}}{a_{\mathrm{CMB}}}\right)^{-2}\left(\frac{a_{\mathrm{CMB}}}{a_0}\right)^{-\frac{3}{2}} \approx 3\times 10^{25}\,, \qquad (11\,\mathrm{p})$$

where we have used that $H(a) \propto a^{-2}$ in the radiation dominated universe and $H(a) \propto a^{-3/2}$ in the matter dominated universe. We therefore obtain $N \approx 2\times 10^{26}$ and so $p \approx 60$. We therefore arrive at the famous result that one needs at least 60 e-folds of inflation in order to solve the horizon problem.

PROBLEM No. 6

(L) In this question you will explore some possibilities of determining the position of a gravitational wave (GW) signal on the sky based on timing data from the three running detectors: LIGO at Hanford & Livingston and Virgo near Pisa. For the purposes of this question, let us assume that the gravitational waves propagate at the speed of light.

The three GW detectors are located at geographical coordinates which we denote by ϕ_i, λ_i (latitude, longitude), where $i = \mathrm{H}, \mathrm{L}, \mathrm{V}$ stand for Hanford, Livingston and Virgo. At this point we leave the values of these coordinates unspecified. Let us denote the radius of the Earth by R.

a) Express the distance d_{ij} between detectors i and j in terms of ϕ_i, λ_i, ϕ_j, λ_j and R. By distance we mean the length of the straight line segment connecting the detectors (which goes under the terrestrial surface). (15 p)

Denote by Δt_{ij} the detection time difference for the detectors i and j. The value of Δt_{ij} will typically be of the order of milliseconds. The knowledge of Δt_{ij} for a single pair of detectors will restrict the locus of possible positions of the GW source on the sky to a circle. Note that in general this will *not* be a great circle.

Initially, let us assume that the detection event occurred at 00:00 of Greenwich sidereal time.

b) Given the knowledge of Δt_{ij} for a single pair of detectors i, j, determine the circle on which the possible positions of the GW source lie. You should answer this question by specifying the equatorial coordinates (right-ascension and declination) of the centre of this circle and its angular radius. Express your answer in terms of Δt_{ij}, ϕ_i, λ_i, ϕ_j, λ_j and R. (20 p)

Let us now assume that the detection occurred at some general Greenwich sidereal time Θ.

c) Redo part b) assuming general Greenwich sidereal time Θ of the detection. Express your answer in terms of Θ, Δt_{ij}, ϕ_i, λ_i, ϕ_j, λ_j and R. (5 p)

We will now consider a particular detection event, namely GW170817, which is widely assumed to be due to a binary neutron star merger. The signal arrived to the Earth at 12:41 GMT on August 17, 2017. It arrived first at Virgo, then 22 milliseconds later at LIGO-Livingston and another 3 milliseconds later at LIGO-Hanford (see arXiv:1710.05833). The geographic coordinates of the three detectors are as follows

$$\begin{aligned} \phi_\mathrm{H} &= 46.4552^\circ\,\mathrm{N} & \lambda_\mathrm{H} &= 119.4075^\circ\,\mathrm{W}\,, \\ \phi_\mathrm{L} &= 30.5021^\circ\,\mathrm{N} & \lambda_\mathrm{L} &= 90.7479^\circ\,\mathrm{W}\,, \\ \phi_\mathrm{V} &= 43.6313^\circ\,\mathrm{N} & \lambda_\mathrm{V} &= 10.5045^\circ\,\mathrm{E}\,. \end{aligned}$$

d) Calculate the value of Θ which corresponds to the detection of GW170817. The equation of time and the R.A. of the Sun for 12:00 GMT of the given date are -4 min and $9^\mathrm{h}48^\mathrm{m}$, respectively. (10 p)

e) Using your results from b) and c), calculate the centres and angular radii of the circles on the sky corresponding to the pairs VH, VL and LH. (10 p)

f) Does the measurement of detection time differences in a system of three detectors determine the position of the source on the sky uniquely? Calculate the possible positions (right-ascension and declination) of the GW170817 source on the sky, which are consistent with the measured detection time differences. Compare your results with the coordinates $\alpha_\mathrm{opt} = 13^\mathrm{h}9^\mathrm{m}48^\mathrm{s}$, $\delta_\mathrm{opt} = -23^\circ22'53''$ of the optical counterpart. *Hint:* Wherever justified, you may replace equation of a circle on a sphere by the equation of a circle on a plane with the same centre and radius. (20 p)

(80 p)

SOLUTION:

a) As it is usual in spherical trigonometry, there are two approaches to this question: we can either work with vectors in Cartesian coordinates (1.) or with spherical triangles (2.).

1. Let us first consider the Cartesian coordinate system xyz where the xy plane coincides with the equatorial plane and the x axis goes in the direction of Greenwich meridian. In such a system, the position of detector i is specified by coordinates

$$(R\cos\phi_i\cos\lambda_i, R\cos\phi_i\sin\lambda_i, R\sin\phi_i)\,. \tag{1}$$

The distance d_{ij} between detectors i and j therefore follows as the magnitude of the relative position vector of the two detectors, that is

$$\begin{aligned} d_{ij} = R\big[&(\cos\phi_i\cos\lambda_i - \cos\phi_j\cos\lambda_j)^2 + (\cos\phi_i\sin\lambda_i - \cos\phi_j\sin\lambda_j)^2 + \\ &+ (\sin\phi_i - \sin\phi_j)^2\big]^{\frac{1}{2}}\,, \end{aligned} \tag{2}$$ (10 p)

which, after some algebra, can be simplified to

$$d_{ij} = R\sqrt{2[1 - \sin\phi_i\sin\phi_j - \cos\phi_i\cos\phi_j\cos(\lambda_i - \lambda_j)]}\,. \tag{3}$$ (5 p)

2. The same answer can be obtained by considering the spherical triangle $\mathsf{D}_i\mathsf{D}_j\mathsf{N}$, where D_i denotes the position of detector i and N denotes the north pole. The distance Δ_{ij} between D_i and D_j along the surface can be expressed using the spherical law of cosines as

$$\cos\frac{\Delta_{ij}}{R} = \sin\phi_i\sin\phi_j + \cos\phi_i\cos\phi_j\cos(\lambda_i - \lambda_j)\,. \tag{4}$$ (5 p)

On the other hand, we have

$$d_{ij} = 2R\cos\frac{\Delta_{ij}}{2R} = 2R\sqrt{\frac{1-\cos\frac{\Delta_{ij}}{R}}{2}}\,. \tag{5}$$ (5 p)

Substituting (4) into (5) indeed yields (3), as it should. (5 p)

b) As it was the case in part a), a solution can be devised using either spherical triangles or vectors. Here we will only consider the vector approach. It is easy to see that the line joining the two detectors can also be characterized as the axis of the circle, which gives the possible positions of the source on the sky. The centre of the circle can therefore be determined as the direction of the relative position vector of the two detectors. Since for the Greenwich sidereal time we have $\Theta = 0^{\mathrm{h}}0^{\mathrm{m}}0^{\mathrm{s}}$, the equatorial and geographic coordinate systems are aligned. That is, the direction to the vernal equinox point ($\alpha = 0°$) coincides with the Greenwich meridian ($\lambda = 0°$). After some algebra, one can show that the equatorial coordinates $\alpha^{\mathrm{c}}_{ij}, \delta^{\mathrm{c}}_{ij}$ of the centre of the circle are determined by the relations

$$\sin\delta^{\mathrm{c}}_{ij} = \frac{R}{d_{ij}}(\sin\phi_i - \sin\phi_j)\,, \tag{6a}$$

$$\tan\alpha^{\mathrm{c}}_{ij} = \frac{\cos\phi_i\sin\lambda_i - \cos\phi_j\sin\lambda_j}{\cos\phi_i\cos\lambda_i - \cos\phi_j\cos\lambda_j}\,. \tag{6b}$$

(15 p)

The angular radius of the circle can be determined from the time difference Δt_{ij} as

$$\cos\rho_{ij} = \frac{c\Delta t_{ij}}{d_{ij}}\,. \tag{7}$$ (5 p)

c) The effect of assuming general Θ is merely to replace $\lambda_i \to \lambda_i + \Theta$, or equivalently $\alpha_{ij} \to \alpha_{ij} - \Theta$ for all i, j in the above formulae. The formulae (7) and (6a) for ρ_{ij} and δ^{c}_{ij} therefore remain unchanged and we obtain

$$(8) \quad \tan(\alpha^{\mathrm{c}}_{ij} - \Theta) = \frac{\cos\phi_i \sin\lambda_i - \cos\phi_j \sin\lambda_j}{\cos\phi_i \cos\lambda_i - \cos\phi_j \cos\lambda_j} \,. \qquad (5\,\mathrm{p})$$

d) We first need to remember that the equation of time gives the discrepancy between the apparent and mean solar time:

$$(9) \quad \text{E.T.} = \text{apparent} - \text{mean} \,. \qquad (10\,\mathrm{p})$$

Further to this, we have $\Theta = \alpha_\odot + \text{apparent}$. Combining these two results, we obtain $\Theta = 10^{\mathrm{h}}25^{\mathrm{m}}$.

e) Substituting numerical values into the formulae derived in b) and c), we obtain

(10a)
$$\alpha^{\mathrm{c}}_{\mathrm{VL}} = 210.2^\circ \qquad \delta^{\mathrm{c}}_{\mathrm{VL}} = +8.45^\circ \qquad \rho_{\mathrm{VL}} = 33.55^\circ \,,$$
(10b)
$$\alpha^{\mathrm{c}}_{\mathrm{VH}} = 191.1^\circ \qquad \delta^{\mathrm{c}}_{\mathrm{VH}} = -1.56^\circ \qquad \rho_{\mathrm{VH}} = 23.56^\circ \,,$$
(10c)
$$\alpha^{\mathrm{c}}_{\mathrm{LH}} = 117.6^\circ \qquad \delta^{\mathrm{c}}_{\mathrm{LH}} = -27.4^\circ \qquad \rho_{\mathrm{LH}} = 72.58^\circ \,.$$

(10 p)

f) Obviously, only two detection time differences are fully independent: the third one always follows automatically. Therefore, only two of the three circles which one can construct give independent information about the position of the source. That is, the three circles will generally intersect at two points. Of course, to find these two points, it is sufficient to only consider intersections of two of the three circles. If we were to work exactly, we would use spherical trigonometry to write down equations for the two circles and solve them simultaneously to obtain the coordinates of intersections. However, such a system of equations can only be solved using advanced numerical methods. On the other hand, we may notice that circles VL and VH are centered near the equator, so, working in radians, their equations are very well approximated by equations of ordinary circles with radii ρ_{ij}. It is then very easy to solve for the coordinates of the two intersections. We have

(11a)
$$\alpha_1 = 13^{\mathrm{h}}18^{\mathrm{m}} \qquad \delta_1 = -23^\circ 18'$$
(11b)
$$\alpha_2 = 11^{\mathrm{h}}52^{\mathrm{m}} \qquad \delta_2 = +17^\circ 35' \,.$$

(20 p)

We therefore observe that the first solution agrees very nicely with the coordinates of the optical counterpart! Note that had we worked exactly, i.e. without approximating the circles on the sphere by planar circles, we would have obtained values $\alpha_1 = 13^{\mathrm{h}}18^{\mathrm{m}}$, $\delta_1 = -23^\circ 24'$. Given the experimental error in determining the detection time differences, such a small difference is completely immaterial.